New Fundamental Wave Equation on Curved Space-Time and its Cosmological Applications

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Flat Space-Time with Minkowski Metric

Elementary particle – its state function transforms as an irrep of the group of the metric - the Poincaré group $P = H_p \otimes_s T(3+1)$

All observers agree on the identity of the elementary particle.

The invariant subgroup T(3+1) of the Poincaré group gives the eigenvalue equations

$$i\partial_\mu \varphi(x) = k_\mu \varphi(x)$$

where $\varphi(x)$ is a scalar wave function and $k^\mu = (\omega, k_x, k_y, k_z)$, $\omega_0^2 = \omega^2 - k^2$ and $k^2 = k^\mu k_\mu$ are labels for the elementary particle

KG and HD Equations in Flat Space-Time

Single application of $i\partial_\mu$ to the eigenvalue equations gives the Klein-Gordon (KG) equation

$$[\partial^\mu \partial_\mu + \omega_0^2]\,\varphi(x) = 0$$

Multiple applications of $i\partial_\mu$ to the eigenvalue equations give two infinite sets of higher-derivative (HD) equations

$$[(\partial^\mu \partial_\mu)^m + \omega_0^{2m}]\,\varphi(x) = 0$$

$$[(\partial^\mu \partial_\mu)^n - \omega_0^{2n}]\,\varphi(x) = 0$$

where *m* and *n* are positive odd and even integers

Generalization of these results to curved space-time of General Relativity (GR).

Curved Space-Time with Arbitrary Metric

Let *M* be a 4D, continuous, pseudo-Riemannian manifold endowed with the metric

$$ds^2 = g^{\mu\nu}(x)\, dx_\mu dx_\nu$$

Coordinatization of *Diff (M)* assigns to diffeomorphism η a set of functions η' such that $x' = x + \eta'$

GR observers - all observers whose coordinate systems are related by the above transformations

Fundamental objects (waves or particles) on *M* have the same properties identified by all GR observers

Fundamental waves in curved space-time

Let $q^{\mu}(x)$ be a vector field defined by

$$i\nabla_{\mu}\varphi(x) = q_{\mu}(x)\varphi(x)$$

where $\varphi(x)$ is a scalar wave function and

$$q^{2}(x) = g^{\mu\nu}(x)q_{\mu}(x)q_{\nu}(x) = \omega_{0}^{2}$$

with ω_0 being the same for all GR observers

HD wave equations in curved space-time

Infinite set $\mathfrak{S}_{\infty}^{+}$ of covariant HD wave equations

$$[(g^{\mu\nu}(x)\nabla_{\mu}\nabla_{\nu})^{m}+\omega_{0}^{2m}]\varphi(x)=0$$

where m = 1, 3, 5, 7, …..

Infinite set $\mathfrak{S}_{\infty}^{-}$ of covariant HD wave equations

$$[(g^{\mu\nu}(x)\nabla_{\mu}\nabla_{\nu})^{n}-\omega_{0}^{2n}]\varphi(x)=0$$

where n = 2, 4, 6, 8, …..

HD in Various Theories

- The gauge-fixing terms in the electroweak theory (Zee 2003)
- Higgs theories (Higgs 1964)
- Modified Einstein's gravitation (Nojiri & Odintsov 2008)
- Quantum theories of gravitation (Weldon 2003)
- General covariant Horava-Lifshitz gravity (Wang at al. 2011)

KG equation in curved space-time

In the infinite set $\mathbb{S}_{\infty}^{+}$ the only fundamental equation is the Klein-Gordon (KG) equation

$$[g^{\mu\nu}(x)\nabla_{\mu}\nabla_{\nu}+\omega_0^2]\varphi(x)=0$$

With the Lagrangian

$$L_{KG}=\frac{1}{2}\left[g^{\mu\nu}(\nabla_{\mu}\varphi)(\nabla_{\nu}\varphi)-\omega_0^2\varphi^2\right]$$

QFT based on this KG equation was already formulated by Fulling (1989) and Wald (1994)

New HD equation in curved space-time

In the infinite set $\mathfrak{S}_{\infty}^{-}$ the only fundamental equation is the fourth-order wave equation

$$[g^{\mu\nu}(x)\, g^{\sigma\rho}(x)\nabla_{\mu}\nabla_{\nu}\nabla_{\sigma}\nabla_{\rho} - \omega_0^4]\,\varphi(x) = 0$$

With the Lagrangian

$$L_{HD} = \frac{1}{2}\left[g^{\mu\nu} g^{\sigma\rho}(\nabla_{\mu}\varphi)(\nabla_{\nu}\varphi)(\nabla_{\sigma}\varphi)(\nabla_{\rho}\varphi) + \omega_0^4\varphi^2\right]$$

The methods originally used by Fulling (1989) and Wald (1994) are generalized to construct a HD QFT

HD QFT in curved space-time

HD QFT was formulated in

(a) globally hyperbolic space-time
by generalizing Wald (1994)

(b) spatially flat Robertson-Walker space-time
by generalizing Fulling (1989)

In both formulations the four solutions are:

$$\omega_{1\pm} = \pm\sqrt{k^2 + \omega_0^2} \quad \text{and} \quad \omega_{2\pm} = \pm\sqrt{k^2 - \omega_0^2}$$

The local eigenvalue equations require that

$\omega_{1\pm}$ and $\omega_{2\pm}$ are real

Matter and Tachyonic Fields

If $\omega_0^2 > 0$ then $\omega_{1\pm}$ are real and describe spinless, massive particles and antiparticles

If $\omega_0^2 > k^2$, then $\omega_{2\pm}$ are complex – to make them real we take $\omega_0 = i\,\tilde{\omega}_0$ and obtain

$$\omega_{2\pm} = \pm\sqrt{k^2 + \tilde{\omega}_0^2}$$

which describe tachyons and anti-tachyons with imaginary mass and superluminal speed

The new HD wave equation describes both matter and tachyonic fields

Scalar Fields in Cosmology

Scalar fields are typically used to explore different inflationary scenarios and to explain Dark Energy

A scalar field used as an inflation field obeys the Klein-Gordon equation (Kolb & Turner 1990, Singh & Singh 2012)

$$\ddot{\phi} + 3H\dot{\phi} + V(\phi) = 0$$

A tachyonic condensate used to explain Dark Energy is represented by a scalar field motivated by string theory

(Bagla et al. 2003 and Calcagni & Liddle 2006)

$$L_{tach} = -V(\phi)\left[1 - \partial_{\mu}\phi\,\partial^{\mu}\phi\right]^{1/2}$$

$$L_{quin} = \frac{1}{2}\partial_{\mu}\phi\,\partial^{\mu}\phi - V(\phi)$$

Cosmological Applications

Scalar fields with $\omega_{1\pm}$ may describe

(i) ordinary matter with spin zero
(kaons, pions and Higgs bosons)

(ii) inflationary fields

(iii) Dark Energy (quintessence)

Scalar fields with $\omega_{2\pm}$ may describe

(i) inflationary fields

(ii) Dark Energy

Main Results

- **New fundamental HD wave equation for a scalar wave function in curved space-time is obtained**
- **HD QFT in globally hyperbolic space-time and in a spatially flat Roberston-Walker space-time are formulated**
- **HD QFT describes scalar fields that may represent matter (particles and anti-particles) and tachyonic fields (tachyons and anti-tachyons)**
- **Scalar fields described by the new HD wave equation may account for ordinary matter (spin zero), an inflation field and Dark Energy**