Cosmology in One Dimension: Correlation, Power Spectra and Void Geometry

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Model Assumptions

- Starting from matter fluctuations, the structure of the universe presents now a hierarchical structure.
- We will use a toy model to follow its evolution after recombination and later, so Newtonian dynamics applies.
- We restrict to 1D gravitating system.
- And use an $N$-body description.
System description 1/2

- Equation of motion

\[ \frac{d^2x}{dt^2} = E(x, t) \]

- Introducing Scaled Space and Time

\[ x = C(t)x' \quad dt = A^2(t)dt' \]

with

\[ C(t) = (\alpha \omega J_0 t)^{2/3} \text{ and } A^2(t) = (\alpha \omega J_0 t) \]

- Transform the equation of motion

\[ \frac{d^2x'}{dt'^2} + \frac{1}{3} \alpha \omega J_0 \frac{dx'}{dt'} - \frac{2}{9} (\alpha \omega J_0)^2 x' = E'(x', t') \]

- The force proportional to \( x' \) will be taken as given by a neutralizing background
System description 2/2

- Here, we consider a planar perturbation. So $i = 1, \infty$ infinite plane sheets.
- For an initially 1-D planar problem we have
  \[
  \frac{d^2x_i''}{dt'^2} + \frac{1}{\sqrt{2}} \omega J_0 \frac{dx_i''}{dt'} - \omega^2 J_0 x_i'' = E_i''
  \]
- For an initially 1-D cylindrical problem we have
  \[
  \frac{d^2x_i''}{dt'^2} + \frac{1}{2} \omega J_0 \frac{dx_i''}{dt'} - \omega^2 J_0 x_i'' = E_i''
  \]
- For an initially 1-D spherical problem we have
  \[
  \frac{d^2x_i''}{dt'^2} + \frac{1}{6} \omega J_0 \frac{dx_i''}{dt'} - \omega^2 J_0 x_i'' = E_i''
  \]
- The friction coefficient $\alpha$ depends on the initial system geometry
- System with periodic boundary conditions
  \rightarrow\text{ periodic potential exactly solved}
For a single particle $i$, the potential $\phi_i(x)$ reads:

$$\phi_i(x) = \frac{\sigma}{2} |x - x_i| \exp(-\kappa |x - x_i|)$$

$$= -\frac{\sigma}{2} \frac{d}{d\kappa} \exp(-\kappa |x - x_i|)$$

For a periodic system of length $2L$, taking into account all replica:

$$\phi_i(x) = -\frac{\sigma}{2} \sum_n \frac{d}{d\kappa} \exp(-\kappa |x - (x_i - 2nL)|)$$

$$\sim \frac{\sigma}{2} \left[ |x - x_i| \exp(-\kappa |x - (x_i - 2nL)|) + \frac{1}{\kappa^2 L} - \frac{1}{2} (x - x_i)^2 \right]$$

Subtracting the neutralizing background contribution is required to obtain a convergent potential

$$\phi_{BG}(x) = -\int_{-\infty}^{\infty} \rho_{BG}(x') |x - x'| \exp(-\kappa |x - x'|) \, dx' = -\frac{\sigma}{2\kappa^2 L}$$

with $\rho_{BG} = -\frac{\sigma}{2L}$. 

The total field for a single particle reads, taking $\kappa \to 0$

$$E_i(x) = -\frac{\sigma}{2} \text{sign}(x - x_i) + \frac{\sigma}{2L} (x - x_i)$$

Summing over all particles, the field for $x$ in the primitive cell reads

$$E(x) = \sum_i E_i(x) = \frac{\sigma}{2} \left[ N_{\text{right}}(x) - N_{\text{left}}(x) + \frac{N}{L} (x - x_c) \right]$$

$x_c = \frac{1}{N} \sum_{i=1}^{N} x_i$ : center of mass, keeps the field constant when a particle leaves the system while another enters from the other side.

B.N. Miller and J.L. Rouet : 2010, PRE 82, 6
Periodic system: symmetry based derivation

- Poisson’s equation for a single particle, background included:
  \[
  \frac{dE_i}{dx} = -\sigma \delta(x - x_i) + \frac{\sigma}{2L}
  \]

- The general solution reads
  \[
  E_i(x) = -\sigma \Gamma(x - x_i) + \frac{\sigma}{2L} x + C
  \]

- Global neutrality on ±L gives \( C = \frac{\sigma}{2} - \frac{\sigma}{2L} x_i \)

\[\Rightarrow\text{ So} \]

\[
E_i(x) = -\frac{\sigma}{2} \text{sign}(x - x_i) + \frac{\sigma}{2L} (x - x_i)
\]
Periodic system: Polarized boundaries

- Another point of view is to write

\[ E_i = \frac{\sigma}{2} \left[ N_{\text{right}}(x) - N_{\text{left}}(x) \right] + E_B \]

where \( E_B \) is the boundary polarization field.

- Particle \( c \) goes out from right and enters left

\[ E_B \]

- Plasma case identical
Numerical Simulation: Initial Condition

- Importance of $P(k)$ to characterize fluctuations
- Power law provides scale-free behavior of primordial Gaussian density fluctuations
- Particles are shifted from their equilibrium position in order to have $P(k) \sim k^n$
- Velocities are connected to the displacement according to the growing mode of the system (trajectories in phase space)
- Figures given for the RF-model and $n = 3$
  - The RF-model is a mathematically consistent 1D model
  - $n = 3$ is the most chaotic choice, corresponds to $n = 1$ in 3D
    (J.A. Peacock, Cosmological Physics)
\( P(k), \ n = 3 \) at \( T = 0 \)

\[ P(k) \sim k^n \text{ with } n = 3 \]

- Require \( n > 0 \) to obtain a hierarchical structure
- Simulation with \( N = 65535 \) particles
A Simulation Result

T = 0.

T = 13.

T = 15.

T = 17.

T = 19.
Multifractal Analysis

- **Box Counting Method**

\[
B(l, q) = \frac{1}{N_{\text{box}}(l)} \sum_{i=1}^{N_{\text{box}}} \mu_i(l)^q \sim l^{\tau_q}
\]

- **Correlation-Integral (or Point-Wise Dimension method)**

\[
Z(l, q) = \frac{1}{N} \sum \text{C}_i(l)^{q-1} \sim l^{\tau_q}
\]

\(N\text{C}_i(l)\) number of particles at a distance \(l\)

- **Density-reconstruction (or \(k\)-neighbor method)**

\[
W(p, \tau_q) = \frac{1}{N} \sum \text{R}_i(p)^{-\tau_q} \sim p^{q-1}
\]

\(R_i\) contains \(k = pN\) points

- \(\tau_q = (q - 1) D_q\) where \(D_q\) is the generalized dimension of order \(q\).
- \(q > 0\) emphasizes high density region, \(q < 0\) emphasizes low density region
- \(D_2\) is the correlation dimension
Simulation results: Box counting

- A scaling range around \( l = 1 \) thanks to the friction

- Two trivial scaling ranges
  - for large \( l \): the slope is 1 due to homogeneity
  - for small \( l \): the slope is 0 due to the discretization (finite number of particles)
Simulation results: Box counting and Correlation-Integral

- Increasing curve!
- For $q \geq 0$ the curve decreases very slightly
- Similar results with Box counting and Correlation-Integral methods
Simulation results : Density-Reconstruction

- Two scaling ranges
- The cutoff is increasing with time
- All curves gather for large $k$ which correspond to homogeneity
$D_q$ : Comparison of the 3 methods

- $D_q$ is Decreasing for the DR method
- Similar results with Box counting and Correlation-Integral method for $q > 0$
\( \tau_q : \) Comparison of the 3 methods

\( T = 16.0 \)

- BC: \(-4.8 < \ln(l) < 2.1\)
- IC: \(-2.7 < \ln(l) < .7\)
- \(K-n\): \(3 < k < 300\)

\[ \tau_q = (q - 1) D_q \]

- The curve \( \tau_q \) suggests a Bi-fractal

Correlation : Time evolution

The slope $n$ gives $D_2 = 1 + n$

For $T = 16$ $n = -0.58$ for $D_2 = 0.40$
Power spectrum evolution

Transition between linear and non-linear regime:

\[ k_c(t) \sim \exp\left(-rt/(n_l - n)\right) \]

with \( r = \gamma(-1 + \sqrt{1 + 4/\gamma^2}) \)

Conclusion

- A 1D toy model including expansion and gravitation
- 1D models allow
  - to use the gravitational field without cutoff
  - to deal with a high number of particles (here 65535)
- Show a hierarchical formation structure in $\mu$-space
- The analysis is performed on the projection in configuration space
- Using large data sets, robust scaling regimes are observed for both low and high density region
- The apparent fractality that arises in observations is a projection from six dimensions
- Share similar fractal properties with observations and 3D simulations (apparent bifractal geometry)
- This remains true for other models (changing the friction coefficient) and other Initial Conditions
- The time evolution of the scalings (power spectrum, correlation) are consistent with universe observations