Redshift drift and supernovae in inhomogeneous pressure cosmology

Mariusz P. Dąbrowski

Institute of Physics, University of Szczecin, Poland
and
Copernicus Center for Interdisciplinary Studies, Kraków, Poland

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Based upon:

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3. Redshift drift test to discriminate between Stephani and LTB.

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5. Combined tests (central observer): luminosity distance, BAO, shift parameter, drift.

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1. Introduction.

Growing interest in **spherically symmetric** Lemaître-Tolman-Bondi (LTB) inhomogeneous density void models with an alternative explanation of the acceleration of the universe became kind of a new paradigm in cosmology (Célérier 2000; Marra et al. 2007; Uzan et al. 2008 and many others) – commented e.g. by Krasiński, Hellaby, Bolejko, and Célérier (2010) that not only LTB are worth investigating.

Let me keep the side of the latter people and consider **spherically symmetric** Stephani models of the pressure gradient.

In fact, LTB and SS Stephani are **complementary models** of the universe and both can mimic homogeneous dark energy models.

The only **common limit** of both is the isotropic Friedmann. Both are the **simplest** inhomogeneous models (SS).

And there are lots of less symmetric or **purely inhomogeneous** models (Goode (1986), Szafron (1977), Szekeres (1975), Wainwright-Goode (1980), Ruiz-Senovilla (1992) etc.) to investigate - cf. other talks.
Complementary models of the spherically symmetric Universe

**Inhomogeneous density** dust shells (LTB) and **inhomogeneous pressure** (gradient of pressure shells) (SS Stephani):

<table>
<thead>
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<th>pressure</th>
<th>density</th>
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<tr>
<td><strong>FRW</strong></td>
<td>$p = p(t)$</td>
<td>$\varrho = \varrho(t)$</td>
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<tr>
<td><strong>LTB</strong></td>
<td>$p = 0\ (p(t))$</td>
<td>$\varrho = \varrho(t, r)$ - nonuniform</td>
</tr>
<tr>
<td><strong>SS Stephani</strong></td>
<td>$p = p(t, r)$ - nonuniform</td>
<td>$\varrho = \varrho(t)$</td>
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2. What are inhomogeneous pressure (Stephani) universes?

They are the only spherically symmetric solutions of Einstein equations for perfect-fluid energy-momentum tensor \( T^{ab} = (\rho + p)u^a u^b + p g^{ab} \) which are conformally flat and embeddable in a 5-dimensional flat space (H. Stephani Commun. Math. Phys. 4, 167 (1967)). We have

\[
\begin{align*}
\text{ds}^2 &= -\frac{a^2}{\dot{a}^2} \frac{a^2}{V^2} \left[ \left( \frac{V}{a} \right) \cdot \right]^2 dt^2 \\
&\quad + \frac{a^2}{V^2} \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] ,
\end{align*}
\]

(1)

where

\[
V(t, r) = 1 + \frac{1}{4} k(t) r^2 ,
\]

(2)

and \((...)\cdot \equiv \partial / \partial t\). The function \(a(t)\) plays the role of a generalized scale factor, \(k(t)\) has the meaning of a time-dependent "curvature index", and \(r\) is the radial coordinate. Compare: LTB has spatially dependent "curvature index" \(K(r)\).
The energy density and pressure are given by

\[
\rho(t) = 3 \left[ \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k(t)}{a^2(t)} \right], 
\]

(3)

\[
p(t, r) = \rho(t) \left\{ -1 + \frac{1}{3} \frac{\dot{\rho}(t)}{\rho(t)} \left[ \frac{V(t,r)}{a(t)} \right] \right\} \equiv w_{eff}(t, r) \rho(t), 
\]

(4)

and generalize the standard Einstein-Friedmann relations

\[
\rho(t) = 3 \left( \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)} \right), 
\]

(5)

\[
p(t) = - \left( 2 \frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k}{a^2(t)} \right) 
\]

(6)

to inhomogeneous models.
SS Stephani universes

Kinematic characteristic of the model:

\[ u_{a;b} = \frac{1}{3} \Theta h_{ab} - \dot{u}_a u_b , \quad \dot{\mathbf{u}} \equiv (\dot{u}_a \dot{u}^a)^{1/2} \cdot \] (7)

where \( \dot{u} \) is the acceleration scalar and the acceleration vector

\[ \dot{\mathbf{u}}_r = \left\{ \frac{a^2}{\dot{a}^2} \frac{a^2}{V^2} \left[ \left( \frac{V}{a} \right) \cdot \right] \right\}_{,r} \] (8)

while the expansion scalar is the same as in FRW model, i.e.,

\[ \Theta = 3 \frac{\dot{a}}{a} . \] (9)

**Compare:** LTB has non-zero expansion and shear.
Inhomogeneous pressure models - null geodesics

The four-velocity and the acceleration are

\[ u_\tau = - c \frac{1}{V}, \quad \dot{u}_r = - c \frac{V_r}{V}. \quad (10) \]

The components of the vector tangent to zero geodesic are

\[ k^\tau = \frac{V^2}{a}, \quad k^r = \pm \frac{V^2}{a^2} \sqrt{1 - \frac{h^2}{r^2}}, \quad k^\theta = 0, \quad k^\varphi = h \frac{V^2}{a^2 r^2}, \quad (11) \]

where \( h = \text{const.} \), and the plus sign in applies to a ray moving away from the centre, while the minus sign applies to a ray moving towards the centre. The acceleration scalar is

\[ \dot{u} \equiv (\dot{u}_\mu \dot{u}^\mu)^{\frac{1}{2}} = \frac{V_r}{a} = \frac{1}{2} \frac{k(t)}{a(t)} r \quad (12) \]

The farther away from the center at \( r = 0 \), the larger the acceleration.
Global topology still $S^3 \times R$. The models are just specific deformations of the de Sitter hyperboloid near the “neck circle”, but with local topology of the constant time hypersurfaces (index $k(t)$) changing in time.

Usually we cut hyperboloid by either $k = 1$ ($S^3$ topology), $k = 0$ ($R^3$) or $k = -1$ ($H^3$) – here we have “3-in-1” – the Universe may either “open up” or “close down”.

standard Big-Bang singularities $a \to 0$, $\rho \to \infty$, $p \to \infty$ are possible (FRW limit).

Finite Density (FD) singularities of pressure appear at some particular value of a radial coordinate $r$ – in standard FRW cosmology there exist exotic (sudden future) singularities of pressure (SFS) with finite scale factor and energy density – they differ (Dąbrowski 2005).

There is no global equation of state - it changes from shell to shell and on the hypersurfaces $t = \text{const}$. 

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Exact classes of inhomogeneous pressure models ...

... have been found in Dąbrowski (1993). In particular, for the so-called Model II one has

\[
\left( \frac{k}{a} \right)' = 0 \quad \text{i.e.} \quad k(\tau) = -\beta a(\tau)
\]

(13)

with the unit \([\beta] = Mpc^{-1}\).

A subcase of model II (from now on IIA) was proposed by Stelmach and Jakacka (2001) – it assumes that the standard barotropic equation of state

\[
\frac{p(\tau)}{c^2} = w\varrho(\tau)
\]

(14)

at the center of symmetry and no exact form of the scale factor. This assumption gives that

\[
\frac{8\pi G}{3c^2} \varrho(\tau) = C^2(\tau) = \frac{A^2}{a^3(w+1)(\tau)} \quad (A = \text{const.})
\]

(15)

and allows to write a generalized Friedmann equation as
and

\[
\frac{p(\tau)}{c^2} = \left[ w + \frac{\beta}{4}(w + 1)a(\tau)r^2 \right] \varrho(\tau) = \varrho_{eff} \varrho(\tau) .
\] (17)

Similarly as in the Friedmann model, we can define critical density as

\[
\varrho_{cr}(\tau) = \frac{3c^2}{8\pi G} \left( \frac{a_{,\tau}}{a(\tau)} \right)^2
\] (18)

and the density parameter \( \Omega(\tau) = \varrho(\tau)/\varrho_{cr}(\tau) \) which after taking \( \tau = \tau_0 \) gives

\[
1 = \frac{A^2}{H_0^2a^{3(w+1)}(\tau_0)} - \frac{\beta c^2}{H_0^2a_0} \equiv \Omega_0 + \Omega_{inh} ,
\] (19)

and so

\[
\beta = \frac{a_0H_0^2}{c^2} (\Omega_0 - 1) .
\] (20)
Exact inhomogeneous pressure models

In Model IIB the scale factor is of the **dust-like type**

\[ a(t) = \sigma t^{2/3}, \quad k(t) = -\alpha \sigma a(t), \quad (21) \]

\((\alpha = (s/km)^{2/3} Mpc^{-4/3}, \quad \sigma = (km/s)^{2/3} Mpc^{1/3}, \quad [t] = s Mpc/km)\) but the equation of state at the center of symmetry is no longer barotropic:

\[ \rho = p \left( \frac{32\pi^2 G^2}{3\alpha^3 c^8} p^2 - \frac{3}{2} \right) . \quad (22) \]

In the limit of the **inhomogeneity parameter** \(\alpha \to 0\) one obtains the Friedmann universe. FD singularity of pressure is at \(r \to \infty\).
3. Redshift drift test to discriminate between Stephani and LTB.

**Redshift drift (Sandage 1962)** - the idea is to collect data from two light cones separated by 10-20 years to look for a change in redshift of a source as a function of time.

![Diagram of redshift drift](image)

There is a relation between the times of emission of light by the source $\tau_e$ and $\tau_e + \delta \tau_e$ and times of their observation at $\tau_o$ and $\tau_o + \delta \tau_o$:

\[
\int_{\tau_e}^{\tau_o} \frac{d\tau}{a(\tau)} = \int_{\tau_e + \delta \tau_e}^{\tau_o + \delta \tau_o} \frac{d\tau}{a(\tau)},
\]

which for small $\delta \tau_e$ and $\delta \tau_o$ reads as

\[
\frac{\delta \tau_e}{a(\tau_e)} = \frac{\delta \tau_o}{a(\tau_o)}.
\]
For small $\delta \tau_e$ and $\delta \tau_o$ we expand in Taylor series

\[
(u_a k^a)_o = (u_a k^a)(r_0, \tau_0 + \delta \tau_0) = (u_a k^a)(r_0, \tau_0) + \left[ \frac{\partial (u_a k^a)}{\partial \tau} \right]_{(r_0, \tau_0)} \delta \tau_0
\]

\[
(u_a k^a)_e = (u_a k^a)(r_e, \tau_e + \delta \tau_e) = (u_a k^a)(r_e, \tau_e) + \left[ \frac{\partial (u_a k^a)}{\partial \tau} \right]_{(r_e, \tau_e)} \delta \tau_e ,
\]

where for inhomogeneous pressure models the redshift reads as

\[
1 + z = \frac{(u_a k^a)_e}{(u_a k^a)_o} = \frac{\frac{V(t_e, r_e)}{R(t_e)}}{\frac{V(t_0, r_0)}{R(t_0)}}
\]

From the definition of the redshift drift by Sandage (1962):

\[
\delta z = z_e - z_0 = \frac{(u_a k^a)(r_e, \tau_e + \delta \tau_e)}{(u_a k^a)(r_0, \tau_0 + \delta \tau_0)} - \frac{(u_a k^a)(r_e, \tau_e)}{(u_a k^a)(r_0, \tau_0)} ,
\]
Redshift drift in inhomogeneous pressure models.

For general SS Stephani metric we obtain

$$\frac{\partial}{\partial \tau} (u_a k^a) = - \left( \frac{1}{a} \right) \dot{a} - \frac{1}{4} \left( \frac{k}{a} \right) \dot{a} r^2, \quad (26)$$

and

$$\frac{\delta z}{\delta \tau_0} = \left[ \frac{\left( \frac{1}{a} \right) + \frac{1}{4} \left( \frac{k}{a} \right) r^2}{1 + \frac{1}{4} k r^2} \right]_e a(\tau_e) - \left[ \frac{\left( \frac{1}{a} \right) - \frac{1}{4} \left( \frac{k}{a} \right) r^2}{1 + \frac{1}{4} k r^2} \right]_o a(\tau_0)(1 + z) \quad (27)$$

For the model with \((k/a) \dot{a} = 0\) we have

$$\frac{\delta z}{\delta \tau} = - \frac{H_0}{1 + \frac{1}{4} k(\tau_0) r_0^2} \left[ \frac{H_e(z)}{H_0} - (1 + z) \right]. \quad (28)$$

Sandage-Loeb CDM formula for \(\Omega_{inh} \to 0; H_e(z) = H_0(1 + z)^{3/2}, r_0 \to 0.\)
Redshift drift - LTB voids.

- Plots for 3 different LTB void models, $\Lambda$CDM, brane DGP, Cold Dark Matter (CMD) (Quercellini et. al, 2012).

- $\Lambda$CDM – the drift is positive at small redshift, but becomes negative for $z \gtrsim 2$.

- Giant void (LTB) model mimicking dark energy - the drift is always negative.
Redshift drift - inhomogeneous pressure models \((r_0 = 0, w = 0)\).

- \(\Omega_{inh} \text{ small} \) - drift as in LTB and CDM models
- \(\Omega_{inh} \text{ larger} \) - drift as in \(\Lambda\)CDM models (first positive, then negative), e.g. for \(\Omega_{inh} = 0.61\) drift is positive for \(z \in (0, 0.34)\).
- \(\Omega_{inh} \text{ very large} \) - drift positive \((\Omega_{inh} = 0.99 \text{ up to } z = 17; \Omega_{inh} = 1 \text{ (inhomogeneity-domination) } z > 0)\) and \(\frac{\delta z}{\delta t} = H_0 \frac{z}{2}\) which means that the drift grows linearly with redshift.
Redshift drift - observational perspective.

- One is able to **differentiate** between the drift in $\Lambda$CDM models, in LTB models, and in Stephani models - this can be done in future experiments.

- At larger $z > 1.7$ redshifts by **giant telescopes**: European Extremely Large Telescope (E-ELT) with spectrograph CODEX (COsmic Dynamics EXperiment); Thirty Meter Telescope (TMT); Giant Magellan Telescope (GMT).

- At smaller (even $z \sim 0.2$) redshifts by **space-borne gravitational wave interferometers** like DECIGO/BBO (DECi-hertz Interferometer Gravitational Wave Observatory/Big Bang Observer).
4. Off-center observer’s position against Union2 supernovae.

The luminosity distance is given by

\[ d_L = \frac{a_0(1 + z)\hat{r}'}{1 + \frac{\beta}{4}a_0r_0^2}, \tag{29} \]

with an off-center observer placed at \( r_0, \theta_0, \phi_0 \) as meant in the coordinate system \( \{ t, r, \theta, \varphi \} \) of the Stephani metric. More precisely we have

\[ d_L = \frac{(1 + z)}{1 - \frac{a_0H_0^2\Omega_{inh}}{4}r_0^2} \hat{r}'(\Omega_{inh}, w, r_0, \theta_0, \varphi_0, H_0, \hat{\theta}', \hat{\varphi}', z), \tag{30} \]

where

\[ \hat{r}' = \hat{r}'(a) = \frac{1}{H_0} \int_{a_e}^{1} \frac{dx}{\sqrt{(1 - \Omega_{inh})x^{1-3w} + \Omega_{inh}x^3}}, \tag{31} \]

and \( a_e \) is the value of the scale factor at the moment of an emission of the light ray.
Off-center observers

For the redshift one takes

\[ 1 + z = \frac{a_0 (4 - a_e H_0^2 \Omega_{inh} r_e^2)}{a_e (4 - a_0 H_0^2 \Omega_{inh} r_0^2)} , \quad (32) \]

where

\[ r_e^2 = (r_0 \sin \theta_0 \cos \varphi_0 + \hat{r}'(a) \sin \hat{\theta}' \cos \hat{\varphi}')^2 \]
\[ + (r_0 \sin \theta_0 \sin \varphi_0 + \hat{r}'(a) \sin \hat{\theta}' \sin \hat{\varphi}')^2 \]
\[ + (r_0 \cos \theta_0 + \hat{r}'(a) \cos \hat{\theta}' \sin \hat{\varphi}')^2 \quad (33) \]

and \( \hat{\theta}' \) and \( \hat{\varphi}' \) are the coordinates of a supernova as seen by an off-center observer in the sky.

We applied **Union2 557 supernovae data** of Amanullah et al. (2010, ApJ, 716, 712) - we note the courtesy of M. Kowalski and U. Feindt to consult the sample.
Best-fit values: inhomogeneity density $\Omega_{inh} \sim 0.74$, center of symmetry equation of state barotropic index $w \sim 0.031$, off-center observer position $Dist = 270$ Mpc ($\chi^2 = 339$).

An observer cannot be farther away than $\sim 3.5 Gpc$. 
Non-barotropic EOS **limits stronger** the position of an observer.
5. Combined tests (central observer): luminosity distance, BAO, shift parameter, drift.

Preliminary plot (Balcerzak, MPD, Denkiewicz, Polarski - in progress):

Not all contours overlap.
Combined tests - keeping inhomogeneity, new EOS $w_1(a)$ parametrization

We suggest a varying barotropic index $w_1(a)$ which can fit the data is:

$$w_1(a) = w + \frac{w_0}{2} \left( 1 + \tanh[\lambda(a_{tr} - a)] \right). \quad (34)$$

where $w$, $w_0$, $\lambda$, and $a_{tr}$ are constants. Here: $\lambda = 40$, $a_{tr} = 0.08$, $w_0 = 0.1$, $w = -0.1$ and $\Omega_{inh} = 0.68$, $z_{tr} \sim 10.66$. Also, as in the standard case:

$$\varrho(a) = \varrho_0 \exp \left[ -3 \int_{a_0}^a da' \frac{1 + w_1(a')}{a'} \right] \equiv \varrho_0 f(a). \quad (35)$$
**Good news:** we can still have inhomogeneity while $w \rightarrow w(a)$!
Advantages for future work:

The general Stephani metric

\[ ds^2 = -\frac{a^2}{\dot{a}^2 \dot{V}^2} \left[ \left( \frac{V}{a} \right) \right]^2 dt^2 + \frac{a^2}{V^2} \left[ dx^2 + dy^2 + dz^2 \right] + a^2 \dot{V}^2 \left[ \frac{dx}{a} \cdot \frac{dx}{a} \right] dt^2 + \frac{a^2}{V^2} \left[ dx^2 + dy^2 + dz^2 \right] , (36) \]

\[ V(t, x, y, z) = 1 + \frac{1}{4} k(t) \left\{ [x - x_0(t)]^2 + [y - y_0(t)]^2 + [z - z_0(t)]^2 \right\} , \]

with \( x_0, y_0, z_0 \) being arbitrary functions of time is just a generalization of both the FRW and SS Stephani metrics in isotropic coordinates.

- **It has no symmetries** acting on spacetime.
- The center of symmetry **changes its position** from slice to slice.
6. Conclusions

- Viable and complementary to LTB cosmologies which can drive acceleration.
- Clear difference against LTB for redshift drift (can be tested by very large telescopes and GW detectors).
- Comparison with the 557 Union2 supernovae data restricts the position of non-centrally placed observers to be not more than $\sim 3.5\text{Gpc}$ away from the center.
- Stephani model fits well the data for SNIa, redshift drift, shift parameter, and BAO provided a specific parametrization for $w = w(a)$ is applied.
- Can even model total spacetime inhomogeneity.