Energy Conditions and DK Stability Criterion in $f(R, L_m)$ Gravity

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My talk is based on our papers:


Outline

● Introduction
● f (R, Lm) gravity
  ▪ Metric formalism
  ▪ Viability conditions
  ▪ Classification
● Constraint Conditions on f (R, Lm) gravity
  ▪ Energy conditions
  ▪ Dolgov-Kawasaki stability criterion
  ▪ Constraints on a class of generalized f(R) models of gravity (Examples)
● Realization of late-time cosmic accelerated expansion in BBHL model
● Conclusions
Introduction

1. The observation Data

(1) The observed rotation curves of M33 — Dark Mater

(2) The observations of SNIa in 1998
Cosmic Acceleration — Dark Energy

\[ v(r) = \sqrt{\frac{GM(r)}{r}} \]

Fig. 1. The observed rotation curve of the dwarf spiral galaxy M33 extends considerably beyond its optical image (shown superimposed); from Roy [69].
2 Explanations to observations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \equiv G_{\mu\nu} = 8\pi G T_{\mu\nu} \]
Introduction

Modified Gravity

- $f(R, L_m), f(G), f(T)...$
- Scalar-tensor theory
- TeVeS theory
- Branelworld gravity
- Ghost condensation
- Horava-Lifshitz gravity
- MOND
  …
**Introduction**

$f (R, L_m)$ theories of gravity have been employed in attempts to explain

1. the present acceleration of the universe;
2. the rotation curves of galaxies and clusters.


**Our work:**
focus on $f (R, L_m)$ to explain the cosmic acceleration
f (R, Lm) gravity

Metric formalism

\[ S = \int f(R, L_m) \sqrt{-g} d^4x, \]


Where \( f(R, L_m) \) is an arbitrary function of the Ricci scalar curvature \( R \) and the Lagrangian density of matter \( L_m \).

The field equations:

\[ \frac{1}{2} F_{L_m} T_{\mu \nu} = F_R R_{\mu \nu} + (g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu) F_R - \frac{1}{2} g_{\mu \nu} [f(R, L_m) - F_{L_m} L_m], \]

where \( \Box = g^{\mu \nu} \nabla_\mu \nabla_\nu \), \( F_R = \frac{\partial f(R, L_m)}{\partial R} \), \( F_{L_m} = \frac{\partial f(R, L_m)}{\partial L_m} \) and the energy-momentum tensor of matter \( T_{\mu \nu} \) is defined as:

\[ T_{\mu \nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu \nu}}. \]
\( f (R, Lm) \) gravity

**Viability conditions**

- must have the correct cosmological dynamics and the evolution of cosmological perturbations;

- must satisfy the stability criterion in order for the gravity model to be theoretical consistent and compatible with experiment;

- must have the correct weak-field limit at both the Newtonian and post-Newtonian levels, i.e., one that is compatible with the available Solar System experiments.

**Refs.**

T. P. Sotiriou, Gen. Rel. Grav. 38, 1407 (2006);
S. 'i. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011);
B. Jain and J. Khoury, Annals Phys. 325, 1479 (2010);
A. de la Cruz-Dombriz, A. Dobado et al., Phys. Rev. D 77, 123515 (2008);
A. de la Cruz-Dombriz, A. Dobado et al., Phys. Rev. Lett. 103 179001 (2009);
M. Abdelwahab, A. Abebe et al., Class. Quant. Grav. 29 135011 (2012);
\( f(R, L_m) \) gravity

**Classification**

From the point of view of curvature-matter coupling

1. Pure \( f(R) \) without curvature-matter coupling \( \rightarrow \) standard \( f(R) \) gravity

   \[
   S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M, \quad f(R) = R \rightarrow \text{GR}
   \]

2. \( f(R, L_m) \) with curvature-matter coupling

   (1) with non-minimal coupling \( \rightarrow \) BBHL model
   
   \[
   S = \int \left\{ \frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] L_m \right\} \sqrt{-g} d^4x
   \]
   
   BBHL, *PRD*75, 104016(2007)

   (2) with arbitrary coupling


   \[
   S = \int \left[ \frac{1}{2} f_1(R) + G(L_m) f_2(R) \right] \sqrt{-g} d^4x,
   \]

$f(R, L_m)$ gravity

\[ f(R, L_m) = f_1(R) + G(L_m)f_2(R) \]

**Gf(R)G model**

- Pure $f(R)$, i.e., standard $f(R)$
- BBHL model
- GR

\[ f_1 = f(R), \quad f_2 = 0 \]
\[ f_2(R) \rightarrow 1 + \lambda f_2(R), \quad G(L_m) = L_m \]
\[ f_1(R) = R, \quad f_2 = 0 \]
Constraint Conditions on \( f(R, L_m) \) gravity

As any \( f(R, L_m) \) theory, there are certain conditions which have to be satisfied in order to ensure that the model is viable and physically meaningful. In the following, we shall study the \( f(R, L_m) \) gravity with arbitrary coupling between curvature and matter from the point of view of the

I. Energy conditions

II. Dolgov-Kawasaki stability criterion


Energy conditions in \( f(R) \) gravity

O Bertolami and M C Sequeira, PRD79, 104010(2009)

Energy conditions and stability in \( f(R) \) theories of gravity with non-minimal coupling to matter


Can modified gravity explain accelerated cosmic expansion?
I. Energy conditions (SEC, NEC, DEC, WEC)

A. The Raychaudhuri Equation

The origin of the strong energy condition (SEC) and null energy condition (NEC) is the Raychaudhuri equation together with the requirement that the gravity is attractive for a space-time manifold. Endowed with a metric $g_{\mu \nu}$ (see S.W.Hawking, G.F.R.Ellis, the large scale structure of space-time, 1973).

In the case of a congruence of timelike geodesics defined by the vector field $u^\mu$, the Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma_{\mu \nu} \sigma^{\mu \nu} + \omega_{\mu \nu} \omega^{\mu \nu} - R_{\mu \nu} u^\mu u^\nu.$$
Constraint Conditions on f (R, Lm) gravity

I. Energy conditions

A. The Raychaudhuri Equation

While in the case of a congruence of null geodesics defined by the vector field $k^\mu$, the Raychaudhuri equation is given

$$\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu$$

where $R_{\mu\nu}$, $\theta$, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are the Ricci tensor, the expansion parameter, the shear and the rotation associated with the congruence.

From above expressions, it is clear that the Raychaudhuri equation is purely geometric and independent of the gravity theory.

However, in order to obtain the Ricci tensor $R_{\mu\nu}$ one needs the gravitational field equation. Thus, through the combination the field equation with the Raychaudhuri equation, one can impose constraints on $T_{\mu\nu}$, which is so-called the energy conditions.
I. Energy conditions

A. The Raychaudhuri Equation

By means of the following relations

\[ \sigma^2 \equiv \sigma_{\mu \nu} \sigma^{\mu \nu} \geq 0 \]
\[ \omega_{\mu \nu} = 0 \text{ (hypersurface orthogonal congruence)} \]

The conditions for gravity to remain attractive \((d\theta/d\tau < 0)\) are

**SEC:**

\[ R_{\mu \nu} u^\mu u^\nu \geq 0 \quad \text{SEC}, \]

**NEC:**

\[ R_{\mu \nu} k^\mu k^\nu \geq 0 \quad \text{NEC}. \]

The FRW metric is chosen as:

\[ ds^2 = dt^2 - a^2(t) \, ds_3^2, \]

**SEC:**

\[ R_{\mu \nu} u^\mu u^\nu = \left( T_{\mu \nu} - \frac{T}{2} g_{\mu \nu} \right) u^\mu u^\nu \geq 0, \quad \rightarrow \quad \rho + 3p \geq 0. \]

**NEC:**

\[ T_{\mu \nu} k^\mu k^\nu \geq 0. \quad \rightarrow \quad \rho + p \geq 0. \]
Constraint Conditions on $f(R, L_m)$ gravity

I. Energy conditions

B. Effective Energy-Momentum Tensor in the generalized $f(R)$ gravity ($Gf(R)G$)

$$S = \int \left[ \frac{1}{2} f_1(R) + G'(L_m) f_2(R) \right] \sqrt{-g} d^4x,$$


Field equation:

$$F_1(R) R_{\mu\nu} - \frac{1}{2} f_1(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F_1(R) = -2G(L_m)$$

$$F_2(R) R_{\mu\nu} - 2 (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) G(L_m) F_2(R) - f_2(R) [K(L_m) L_m - G(L_m)] g_{\mu\nu} + f_2(R) K(L_m) T_{\mu\nu},$$

where $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$, $F_i(R) = df_i(R)/dR$ $(i = 1, 2)$ and $K(L_m) = dG(L_m)/dL_m$ respectively. The energy–momentum tensor of matter is defined as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}.$$

In this class of models, the energy–momentum tensor of matter is generally not conserved due to the appearance of an extra force.
Constraint Conditions on f (R, L_m) gravity

I. Energy conditions


B. Effective Energy-Momentum Tensor in the Gf(R)G

Field equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{\text{eff}}_{\mu\nu},$$

where the effective energy-momentum tensor $T^{\text{eff}}_{\mu\nu}$ is defined as follows:

$$T^{\text{eff}}_{\mu\nu} = \frac{1}{f'_1 + 2Gf'_2} \left\{ \frac{1}{2} g_{\mu\nu} \left[ f_1 - \left( f'_1 + 2GF'_2 \right) R \right] 
- \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f'_1 
- 2 \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) Gf'_2 
- f_2 \left( G'L_m - G \right) g_{\mu\nu} + f_2 G'T_{\mu\nu} \right\}.$$ 

Contracting the above equation, we have

$$T^{\text{eff}} = \frac{1}{f'_1 + 2Gf'_2} \left\{ 2 \left[ f_1 - \left( f'_1 + 2Gf'_2 \right) R \right] - 3 \Box f'_1 
- 6 \Box Gf'_2 - 4f_2 \left( G'L_m - G \right) + f_2 G'T \right\},$$
I. Energy conditions

C. Strong and Null Energy Conditions in the $Gf(R)G$

\[ R_{\mu\nu} u^\mu u^\nu \geq 0 \quad \text{SEC} \quad \Rightarrow \quad T_{\mu\nu}^{\text{eff}} u^\mu u^\nu - \frac{1}{2} T^{\text{eff}} \geq 0, \]

(a) SEC:

\[
\begin{align*}
\rho + 3p & - \frac{1}{f_2 G'} \left[ f_1 - \left( f_1' + 2Gf_2' \right) \dot{R} \right] + 3 \frac{f_1''}{f_2 G'} \left( H \dot{R} \dot{G} + \ddot{R} \right) \\
& + 3 \frac{f_1'''}{f_2 G'} \dot{R}^2 + 6 \frac{1}{f_2 G'} \left( G'' \dot{L}_m f_2 + \ddot{L}_m G' f_2' + 2f_2'' \dot{R} G' \dot{L}_m 
+ f_2''' \dot{R}^2 G + f_2'' \ddot{R} G \right) + 6 \frac{H}{f_2 G'} \left( G' \dot{L}_m f_2' + f_2'' \dot{R} G \right) \\
& + \frac{2}{G'} \left( G' \dot{L}_m - G \right) \geq 0,
\end{align*}
\]

where the dot denotes differentiation with respect to cosmic time. This is the SEC in $f(R)$ gravity with arbitrary coupling between matter and geometry.
Constraint Conditions on $f\left(R, L_m\right)$ gravity

I. Energy conditions

C. Strong and Null Energy Conditions in the $Gf(R)G$

The NEC in $f\left(R\right)$ gravity with arbitrary coupling between matter and geometry can be expressed as:

$$R_{\mu\nu}k^\mu k^\nu \geq 0 \quad \text{NEC} \quad \implies \quad T_{\mu\nu}^{\text{eff}}k^\mu k^\nu \geq 0.$$ 

(b) NEC:

$$\rho + p + \left(H \dot{R} + \ddot{R}\right) \frac{f_1''}{f_2 G'} + \frac{f_1'''}{f_2 G'} \dot{R}^2 + \frac{2}{f_2 G'} (G'' \dot{L}_m \dot{f}'_2 + \ddot{L}_m G' f'_2 + 2 f''_2 \ddot{R} G' \dot{L}_m + f'''_2 \dot{R}^2 G + f''_2 \ddot{R} G) - \frac{2H}{f_2 G'} (G' \ddot{L}_m f'_2 + f'''_2 \ddot{R} G) \geq 0.$$
Constraint Conditions on $f(R, L_m)$ gravity

I. Energy conditions

D. Dominant and Weak Energy Conditions in the $G_f(R)G$

Note that the above expressions of the SEC and NEC are directly derived from Raychaudhuri equation. However, equivalent results can be obtained by taking the following transformations. Thus by extending this approach, we will give the DEC and the WEC in generalized $f(R)$ gravity with arbitrary coupling between matter and geometry ($G_f(R)G$) in the following.

\[
\begin{align*}
\rho &\rightarrow \rho^{\text{eff}} \\
p &\rightarrow p^{\text{eff}} \\
\rho^{\text{eff}} + 3p^{\text{eff}} &\geq 0 \\
\rho^{\text{eff}} + p^{\text{eff}} &\geq 0 \\
\rho^{\text{eff}} - p^{\text{eff}} &\geq 0 \\
\rho^{\text{eff}} &\geq 0
\end{align*}
\]

(SEC) (NEC) (SEC) (WEC)
Constraint Conditions on $f (R, L_m)$ gravity

I. Energy conditions

D. Dominant and Weak Energy Conditions in the $Gf(R)G$

$$T_{\mu\nu}^{\text{eff}} = \frac{1}{f_1' + 2Gf_2'} \left\{ \frac{1}{2} g_{\mu\nu} \left[ f_1 - (f_1' + 2Gf_2')R \right] \right.$$

$$- (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_1' - 2(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) Gf_2'$$

$$- f_2 (G'L_m - G) g_{\mu\nu} + f_2 G'T_{\mu\nu} \right\},$$

$$\rho^{\text{eff}} = \frac{1}{f_1' + 2Gf_2'} \left\{ \frac{1}{2} \left[ f_1 - (f_1' + 2Gf_2')R \right] - 3H \dot{R} f_1'' \right.$$

$$- 6H (G'L_m f_2' + f_2'' \dot{R} G) - f_2 (G'L_m - G) + f_2 G' \rho \right\},$$

$$p^{\text{eff}} = \frac{1}{f_1' + 2Gf_2'} \left\{ -\frac{1}{2} \left[ f_1 - (f_1' + 2Gf_2')R \right] + (2H \dot{R} + \ddot{R}) f_1'' + f_1''' \dddot{R}^2 \right.$$

$$+ 2(G'' L_m f_2' + L_m G' f_2' + 2f_2'' \dot{R} G' L_m + f_2'''' \dddot{R}^2 G + f_2'' \dddot{R} G)$$

$$4H (G'L_m f_2' + f_2'' \dot{R} G) + f_2 (G'L_m - G) + f_2 G' \rho \right\}.$$
Constraint Conditions on f (R, Lm) gravity

I. Energy conditions
   D. Dominant and Weak Energy Conditions in the Gf (R)G

Dominant energy condition (DEC) in the Gf(R)G:

\[
\rho - p + \frac{1}{f_{2 G'}} \left[ f_1 - (f'_1 + 2G f''_2) R \right] - (5H \dot{R} + \ddot{R}) \frac{f''}{f_{2 G'}} \nonumber \\
- \frac{f''}{f_{2 G'}} \ddot{R}^2 - \frac{2}{f_{2 G'}} (G'' L_m^2 f'_2 + L_m G' f'_2 + 2 f''_2 \dot{R} G' \dot{L}_m) \nonumber \\
+ f''_2 \dot{R}^2 G + f''_2 \ddot{R} G) - \frac{10H}{f_{2 G'}} (G' L_m f'_2 + f''_2 \dot{R} G) \nonumber \\
- \frac{2}{G'} (G' L_m - G) \geq 0,
\]

Weak energy condition (WEC) in the Gf(R)G:

\[
\rho + \frac{1}{2f_{2 G'}} \left[ f_1 - (f'_1 + 2G f''_2) R \right] - 3H \dot{R} \frac{f''}{f_{2 G'}} - 6H \frac{1}{f_{2 G'}} (G' L_m f'_2 + f''_2 \dot{R} G) - \frac{1}{G'} (G' L_m - G) \geq 0.
\]
Constraint Conditions on f (R, Lm) gravity

I. Energy conditions

The comments on the above energy conditions:

1 These energy conditions can degenerate to the ones in the special cases including the non-minimal coupled and pure f (R) gravities as well as GR.

Energy conditions in f(R) gravity
Energy conditions and stability in f(R) theories of gravity with non-minimal coupling to matter

2 The energy conditions have been extended to f (R, Lm) and f (G)

Refs: J Wang and K Liao, Class Quantum Grav. 29, 215016(2013)
Energy conditions in f (R, Lm) gravity;
Modified f (G) gravity models with curvature-matter coupling;
Y Y Zhao, Y B Wu et al., submitted to Phys. Lett. B (2013)
Constraint Conditions on \( f(R, L_m) \) gravity

I. Energy conditions

The comments on the above energy conditions:

3 Challenge

It is worth stressing that as pointed in Ref. [JCAP 1307(2013) 009]

- these energy conditions remains doubtful since
  
  there is no natural motivation but only an analogy with GR. This extension is in fact only motivated when the new terms appearing in the field equations are identified with physical fields. Nevertheless, these new terms may be understood as possessing only a geometrical meaning. Thus, there is no reason to assume any energy conditions on these terms.

JCAP 1307(2013) 009

On the non-attractive character of gravity in \( f(R) \) theories

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I. Energy conditions

The comments on the above energy conditions:

3 Challenge

- SEC in GR: \( R_{\mu\nu} u^\mu u^\nu \geq 0 \)

which may be interpreted, because of asserting a non-positive contribution to Raychaudhuri equation, as a manifestation of the attractive character of gravity. It

But, it may be not satisfied for the modified gravity to account for the present accelerated expansion of the universe.

Therefore, if those extended energy conditions hold, the present accelerated expansion of the Universe cannot be accommodated.

- Our present work:
  Discussing the non-attractive character of gravity in f(R,Lm)
A viable modified gravity model must pass Newton law, solar system test and instability conditions.

There are in principle several kinds of instabilities to consider. Dolgov–Kawasaki instability is one of them.


Can modified gravity explain accelerated cosmic expansion?

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Constraint Conditions on $f(R, L_m)$ gravity

II. Dolgov-Kawasaki stability criterion in the $f(R,L_m)$

J Wang, Y B Wu et al., *Phys. Lett. B* 689 (2010);

Y Y Zhao, Y B Wu et al., submitted to *Phys. Lett. B* (2013)

Constraints of energy conditions and DK instability criterion on $f(R, L_m)$ gravity models

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Constraint Conditions on $f(R, L_m)$ gravity

II. Dolgov-Kawasaki stability criterion in the $f(R, L_m)$

We consider a type of specific $f(R, L_m)$ models as

$$f(R, L_m) = \frac{1}{2}f_1(R) + f_2(R)G_2(L_m) + G_1(L_m),$$

where $f_i(R)(i = 1, 2)$ and $G_j(L_m)(j = 1, 2)$ are arbitrary functions of the Ricci scalar and Lagrangian density of matter, respectively.

The trace of the field equation

$$[f'_1(R) + 2f'_2(R)G_2(L_m)]R + 3\nabla^\mu\nabla_\mu[f'_1(R) + 2f'_2(R)G_2(L_m)]$$

$$-2[f_1(R) + 2f_2(R)G_2(L_m) + 2G_1(L_m)] + 4[f_2(R)G'_2(L_m) + G'_1(L_m)]L_m$$

$$= [f_2(R)G'_2(L_m) + G'_1(L_m)]T.$$
Constraint Conditions on $f(R, L_m)$ gravity

II. Dolgov-Kawasaki stability criterion in the $f(R,L_m)$

In the perturbations

\[
\begin{align*}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \\
R &= R_0 + R_1, \ T = T_0 + T_1, \ L = L_0 + L_1, \\
f_1(R) &= R + \epsilon \varphi(R), \\
f_1(R) &= R_0 + R_1 + \epsilon \varphi(R_0) + \epsilon \varphi'(R_0)R_1 + ... , \\
f_1'(R) &= 1 + \epsilon \varphi'(R_0) + \epsilon \varphi''(R_0)R_1 + \cdots
\end{align*}
\]

where $\eta$ is the Minkowski metric. Actually this is a local expansion over small spacetime regions that are locally flat. Accordingly, $f_1(R) = R_0 + R_1 + \epsilon \varphi(R_0) + \epsilon \varphi'(R_0)R_1 + ...$, $f_1'(R) = 1 + \epsilon \varphi'(R_0) + \epsilon \varphi''(R_0)R_1 + ...$ and the linearized version of the trace equation
The linearized version of the trace equation in the perturbations:

\[
3[\epsilon \varphi''(R_0) + 2 f''_2(R_0) G_2(L_0)] \Box R_1 + [-1 - \epsilon \varphi'(R_0) - 2 f'_2(R_0) G_2(L_0) + \epsilon \varphi''(R_0) R_0 + 2 f''_2(R_0) G_2(L_0) R_0 + 4 f'_2(R_0) G'_2(L_0) L_0 - f'_2(R_0) G'_2(L_0) T_0 + 6 f''_2(R_0) \Box G_2(L_0)] R_1 = -2 f'_2(R_0) G'_2(L_0) R_0 L_1 - 4 [f_2(R_0) G''_2(L_0) + G'_1(L_0)] T_0 L_1 + [f_2(R_0) G'_2(L_0) + G'_1(L_0)] T_1.
\]

The above equation can be rewritten as:

\[
\ddot{R}_1 - \nabla^2 R_1 + m^2_{\text{eff}} R_1 = \left\{3[\epsilon \varphi''(R_0) + 2 f''_2(R_0) G_2(L_0)]\right\}^{-1} \left\{-2 f'_2(R_0) \times G'_2(L_0) R_0 L_1 - 4 [f_2(R_0) G''_2(L_0) + G''_1(L_0)] L_0 L_1 - 6 f'_2(R_0) G'_2(L_0) \Box L_1 + [f_2(R_0) G''_2(L_0) + G''_1(L_0)] T_0 L_1 + [f_2(R_0) G'_2(L_0) + G'_1(L_0)] T_1\right\},
\]
II. Dolgov-Kawasaki stability criterion in the $f(R, L_m)$

By further calculation, the effective mass $m_{eff}$ of the dynamical degree of freedom $R_1$ can be given as

$$m_{eff}^2 = \left\{3[\epsilon \varphi''(R_0) + 2f''_2(R_0)G_2(L_0)]\right\}^{-1}[-1 - \epsilon \varphi'(R_0) - 2f'_2(R_0)G_2(L_0) + \epsilon \varphi''(R_0)R_0 - f'_2(R_0)G'_2(L_0)T_0 + 6f''_2(R_0)\Box G_2(L_0)].$$

The dominant term on the right side is $\{3[\epsilon \varphi''(R_0) + 2f''_2(R_0)G_2(L_0)]\}^{-1}$. Because the stability requires that the effective mass squared must be non-negative, we have

$$f''_1(R) + 2f''_2(R)G_2(L_m) \geq 0,$$

which is just the Dolgov-Kawasaki stability criterion in the $f(R, L_m)$. And it is the same as the one in the generalized $f(R)$ gravity with arbitrary curvature-matter coupling.
Constraint Conditions on $f(R, L_m)$ gravity

II. Dolgov-Kawasaki stability criterion in the $f(R, L_m)$

$$f(R, L_m) = \frac{1}{2}f_1(R) + f_2(R)G_2(L_m) + G_1(L_m),$$

$$f'_1(R) + 2f'_2(R)G_2(L_m) \geq 0,$$

It follows that DK criterion in our models is the same as the one in the generalized $f(R)$ gravity with arbitrary geometry-matter coupling $(G f(R) G)$, which indicates that DK criterion is only related to the term of coupling, and irrespective of $G_1(L_m)$.

It is worth stressing that if let $f_2(R) = 1 + \lambda f_2(R), G_2(L_m) = L_m, G_1(L_m) = 0$, the DK criterion in $f(R)$ gravity with non-minimal coupling will be obtained. Furthermore, if taking $f_2(R) = 1, G_2(L_m) = L_m, G_1(L_m) = 0$, the DK criterion is consistent with the one in the pure $f(R)$ gravity.
Constraint Conditions on $f(R, L_m)$ gravity

III. Constraints on a class of $Gf(R)G$

$$f_1(R) = R + \epsilon R^n, \quad f_2(R) = \alpha R^m.$$ 

In the FRW cosmology, the energy conditions and the Dolgov–Kawasaki instability criterion can be unifiedly expressed as

$$\frac{\dot{\epsilon}}{\dot{\alpha}} \left| \frac{R^n}{R^m G'(L_m)} \right| \left\{ \frac{2\dot{\alpha}}{\dot{\epsilon}} \left[ G(L_m)C_m + A \right] \right\} R^{m-n} + C_n \geq B,$$

where $A, B$ and $C_{m,n}$ depend on the energy condition under study and we take $\dot{\epsilon} = (-1)^n \epsilon$ and $\dot{\alpha} = (-1)^n \alpha$ due to the fact that for a FRW metric one has $R < 0$. 

Constraint Conditions on $f(R, L_m)$ gravity

III. Constraints on a class of $G_f(R)G$

(1) Constraints of energy conditions

For the SEC

$$A^{SEC} = G'(L_m)[L_m + 3mR^{-1}(\ddot{L}_m + H\dot{L}_m) + 6mL_m\dot{R}R^{-2}(m - 1)] + 3mL_m^2 G''(L_m)R^{-1},$$

$$B^{SEC} = -(\rho + 3\rho),$$

$$C_n^{SEC} = (n - 1)[3\ddot{R}nR^{-2} + 1 + 3HnR^{-2}\dot{R} + 3nR^{-3}\dot{R}^2(n - 2)].$$
Constraint Conditions on $f (R, L_m)$ gravity

III. Constraints on a class of $Gf(R)G$

(1) Constraints of energy conditions

For the NEC, one obtains

\[
A^{\text{NEC}} = [m \dddot{L}_m G'(L_m) R^{-1} - Hm \ddot{L}_m G'(L_m) R^{-1} \\
+ 2m \dot{L}_m \dddot{G}'(L_m) R^{-2} (m - 1) + m \dddot{L}_m G''(L_m) R^{-1},
\]

\[
B^{\text{NEC}} = -(\rho + p),
\]

\[
C_n^{\text{NEC}} = (n - 1)n[\dddot{R} R^{-2} + H R^{-2} \dddot{R} + R^{-3} \dddot{R}^2 (n - 2)].
\]
III. Constraints on a class of Gf(R)G

(1) Constraints of energy conditions

For the DEC, one has

$$A^{DEC} = G'(L_m) \left[ -L_m - m\dot{L}_m R^{-1} - 5Hm\dot{L}_m R^{-1} 
+ (1 - m)2m\dot{L}_m \dot{R} R^{-2} \right] - m\dot{L}_m^2 G''(L_m) R^{-1},$$

$$B^{DEC} = -(\rho - p),$$

$$C^{DEC}_n = (1 - n)[\ddot{R}nR^{-2} + 1 + 5HnR^{-2} \dot{R} + nR^{-3} \dot{R}^2 (n - 2)].$$
III. Constraints on a class of $Gf(R)G$

(1) Constraints of energy conditions

For the WEC, one gets

$A^{WEC} = -L_m - 6H_m L_m R^{-1},$

$B^{WEC} = -\rho,$

$C_n^{WEC} = (1 - n)(\frac{1}{2} + 3H_n R^{-2} \dot{R}).$
Constraint Conditions on f (R, Lm) gravity

III. Constraints on a class of Gf(R)G

(1) Constraints of energy conditions

For the WEC, espantially when taking

\[ f_1(R) = R \]
\[ f_2(R) = \alpha R^n \]

For this particular model, the WEC is

\[ 0.3Bn^2 - 0.3n(1 + B) + 1 \geq 0, \]

where \( B = (j - q - 2)/(1 - q)^2 \), and the definitions of the deceleration (\( q \)) and the jerk (\( j \)) are

\[ q = -\frac{1}{H^2} \frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3} \frac{\dddot{a}}{a} \]
III. Constraints on a class of Gf(R)G

(1) Constraints of energy conditions

For this particular model, the WEC is

\[ 0.3Bn^2 - 0.3n(1 + B) + 1 \geq 0, \]

The coefficient \( \alpha \) is arbitrary and the value of the index \( n \) depends on \( B \).

Taking \( q_0 = -0.81 \pm 0.14, \ j_0 = 2.16^{+0.81}_{-0.75} \), then \( B \) is \( 0.03 \leq B_0 \leq 0.5 \).

When \( 0.03 \leq B \leq \frac{17 - 2\sqrt{70}}{3} \),

\[ 7.36577 \leq n_+ \leq 30.716 \text{ and } 3.61737 \leq n_- \leq 5.26214. \]

When \( \frac{17 - 2\sqrt{70}}{3} < B \leq 0.5 \), the index \( n \) can be taken as any real number.
Constraint Conditions on f (R, Lm) gravity

III. Constraints on a class of Gf(R)G

(2) Constraints of the Dolgov–Kawasaki instability criterion

\[ f_1(R) = R + \epsilon R^n, \]
\[ f_2(R) = \alpha R^m. \]

\[
\frac{\hat{\epsilon} |R|^n}{\hat{\alpha} |R|^m G'(L_m)} \left\{ \frac{2\hat{\alpha}}{\hat{\epsilon}} [G(L_m) c_m + A] |R|^{m-n} + c_n \right\} \geq B,
\]

\[ A^{DK} = 0, \]
\[ B^{DK} = 0, \]
\[ C_n^{DK} = n(n - 1)\hat{\alpha} |R|^m G'(L_m). \]
Realization of late-time cosmic accelerated expansion in BBHL


Model 1

\[ f_1(R) = R, \quad f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1}, \]

where \( c_1 \) and \( c_2 \) are constants.

\[ L_m = -\rho = -\rho_0 a^{-3(1+w)}, \]

where \( w \) is the equation of state of perfect fluid and is assumed to be a constant.

\[ T_{\mu\nu} = (\rho + p) U_\mu U_\nu + pg_{\mu\nu}. \]

FRW metric is chosen as

\[ ds^2 = -dt^2 + a^2(t) dx_3^2, \]

where \( a(t) \) is the scale factor and \( dx_3^2 \) contains the spatial part of the metric.
Realization of late-time cosmic accelerated expansion in BBHL

**Model 1**

\[
f_1(R) = R, \quad f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1},
\]

\[
3H^2 = -\rho_0 a^{-3(1+w)}[1 + \frac{6^n c_1 n \lambda (H^2 + \dot{H})(2H^2 + \dot{H})^{-1+n}}{[1 + 6^n c_2 (2H^2 + \dot{H})^n]^2} + \frac{6^n c_1 \lambda (2H^2 + \dot{H})^n}{1 + 6^n c_2 (2H^2 + \dot{H})^n}].
\]

If \( a = a_0 t^p \), we have

\[
H = \frac{\dot{a}(t)}{a(t)} = \frac{p}{t},
\]

\[
R = 6(2H^2 + \dot{H}) = \frac{6p}{t^2} (2p - 1),
\]

\[
q = -\frac{1}{H^2} \frac{\ddot{a}}{a} = \frac{1}{p} - 1.
\]
Realization of late-time cosmic accelerated expansion in BBHL

**Model 1**

\[ f_1(R) = R, \quad f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1}, \]

By calculations and analysis, the relationship among \( p \), \( w \) and \( n \), condition and candidate for late-time cosmic accelerated expansion are shown in Table 1.

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Condition</th>
<th>Candidate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = \frac{2(1-n)}{3(1+w)} )</td>
<td>( n \geq 0 ) and ( n \neq 1 )</td>
<td>The effective quintessence</td>
</tr>
<tr>
<td>( p = \frac{2(1-2n)}{3(1+w)} )</td>
<td>( n \geq 0 ) and ( n \neq 1/2 )</td>
<td>( 0 \leq n &lt; 1 )</td>
</tr>
<tr>
<td>( p = \frac{2}{3(1+w)} )</td>
<td>( w &lt; -1/3 )</td>
<td>All ( n )</td>
</tr>
</tbody>
</table>

Table 1. The relationship among \( p \), \( w \) and \( n \), condition and candidate for late-time cosmic accelerated expansion in case \( f_1(R) = R, f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1} \).
Realization of late-time cosmic accelerated expansion in BBHL

Model 2

\[ f_1(R) = R + \frac{c_1 R^n}{c_2 R^n + 1}, \quad f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1}, \]

where \( c_1 \) and \( c_2 \) are constants.

the FRW equation can be given as:

\[
\frac{1}{2}[12H^2 + 6\dot{H} + \frac{6^n c_1 (2H^2 + \dot{H})^n}{1 + 6^n c_2 (2H^2 + \dot{H})^n} - 6(H^2 + \dot{H})(1 + \frac{6^{-1+n} c_1 n (2H^2 + \dot{H})^{-1+n}}{[1 + 6^n c_2 (2H^2 + \dot{H})^n]^2})] =
\]

\[-\rho_0 a^{-3(1+w)}[1 + \frac{6^n c_1 n \lambda (H^2 + \dot{H}) (2H^2 + \dot{H})^{-1+n}}{[1 + 6^n c_2 (2H^2 + \dot{H})^n]^2} + \frac{6^n c_1 \lambda (2H^2 + \dot{H})^n}{1 + 6^n c_2 (2H^2 + \dot{H})^n}].\]
Model 2

\[ f_1(R) = R + \frac{c_1 R^n}{c_2 R^n + 1}, \quad f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1}, \]

If \( a = a_0 t^p \), we have

\[
H = \frac{\dot{a}(t)}{a(t)} = \frac{p}{t}, \\
R = 6(2H^2 + \dot{H}) = \frac{6p}{t^2}(2p - 1), \\
q = -\frac{1}{H^2} \frac{\ddot{a}}{a} = \frac{1}{p} - 1.
\]
Realization of late-time cosmic accelerated expansion in BBHL

**Model 2**

\[
f_1(R) = R + \frac{c_1 R^n}{c_2 R^n + 1}, \quad f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1},
\]

The corresponding relationship among \(p, w\) and \(n\), condition and candidate for late-time cosmic accelerated expansion are shown in Table 2.

<table>
<thead>
<tr>
<th>Relationship</th>
<th>Condition</th>
<th>Candidate</th>
<th>Candidate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p = \frac{2}{3(1+w)})</td>
<td>(w &lt; -1/3)</td>
<td>All (n)</td>
<td>All (n)</td>
</tr>
<tr>
<td>(p = \frac{4n+2}{3(1+w)})</td>
<td>(n \leq 0) and (n \neq -1/2)</td>
<td>(-1/2 &lt; n \leq 0)</td>
<td>(n &lt; -1/2)</td>
</tr>
<tr>
<td>(p = \frac{2(n+1)}{3(1+w)})</td>
<td>(n \leq 0) and (n \neq -1)</td>
<td>(-1 &lt; n \leq 0)</td>
<td>(n &lt; -1)</td>
</tr>
<tr>
<td>(p = \frac{2n}{3(1+w)})</td>
<td>(n \leq 1) and (n \neq 0)</td>
<td>(0 &lt; n \leq 1)</td>
<td>(n &lt; 0)</td>
</tr>
<tr>
<td>(p = \frac{4n}{3(1+w)})</td>
<td>(n \leq 1/2) and (n \neq 0)</td>
<td>(0 &lt; n \leq 1/2)</td>
<td>(n &lt; 0)</td>
</tr>
</tbody>
</table>

Table 2. The relationship among \(p, w\) and \(n\), condition and candidate for late-time cosmic accelerated expansion in case \(f_1(R) = R + \frac{c_1 R^n}{c_2 R^n + 1}; f_2(R) = \frac{c_1 R^n}{c_2 R^n + 1}.\)
On the consistency of non-minimally coupled $f(R)$ gravity

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(Dated: August 16, 2013)

Theories with a non-minimal coupling between the space-time curvature and matter fields introduce an extra force due to the non-conservation of the matter energy momentum. In the present work the theoretical consistency of such couplings is studied using a scalar field Lagrangian to model the matter content. The conditions that the coupling does not introduce ghosts, classical instabilities or superluminal propagation of perturbations are derived. These consistency conditions are then employed to rule out or severely restrict the forms of the non-minimal coupling functions considered in the previous literature. For example, a power-law coupling is viable only for sublinear positive power of the curvature scalar.
It is pointed out in PRD76(2013), gr-qc/1308.3401v1

Because of their applications to dark energy and inflation, these theories have been well studied and several results have already been discovered. In particular we are interested in the stability conditions one must impose on the function \( p \) in order for the model to be physically viable. These can be obtained requiring the positivity of both the energy density of the scalar field and the speed of sound at which the perturbations propagate in the scalar fluid. The required conditions are given by \([54, 55]\)

\[
\begin{align*}
\epsilon(\phi) &= 2X_\phi p_{,x_\phi} - p > 0, \\
c_s^2(\phi) &= \frac{p_{,x_\phi}}{\epsilon_{,x_\phi}} = \frac{p_{,x_\phi}}{2X_\phi p_{,x_\phi} x_\phi + p_{,x_\phi}} \geq 0,
\end{align*}
\] (19, 20)

where \( ,x_\phi \) denotes differentiation with respect to \( X_\phi \). Here \( \epsilon(\phi) \) is the energy density of the scalar field, while \( c_s(\phi) \) is the speed of sound. Conditions (19) and (20) have to be satisfied by every physically viable model. If they do not hold the theory cannot be employed at macroscopic scales. If we further require that the perturbations do not propagate faster than the speed of light we should impose the further condition that \( c_s(\phi) \leq 1 \).

\[
\epsilon > 0 \quad \text{and} \quad 1 \geq c_s^2 \geq 0,
\]

which can be computed directly from the Jordan frame avoiding the passage to the Einstein frame. As we will see, once a particular \( f(R) \) model is chosen in action \( \Pi \), the conditions \([26]\) will permit to constrain the free parameters of the theory.
Realization of late-time cosmic accelerated expansion in BBHL

CONSISTENCY OF SPECIFIC MODELS

A. \( f_1(R) \propto R \) and \( f_2(R) \propto R^\gamma \)
\[ 0 \leq c_s^2 \leq 1 \iff 0 \leq \gamma \leq 1. \]

B. \( f_1(R) \propto R \) and \( f_2(R) = 1 + (R/R_0)^\gamma \)
\[ 0 \leq c_s^2 \leq 1 \iff 0 \leq \gamma \leq 1. \]

C. \( f_1(R) \propto R \) and \( f_2(R) \propto e^{\lambda R} \)

D. \( f_1(R) = \frac{R}{16\pi G} + A R^n \) and \( f_2(R) \propto R^n \)
\[ 0 \leq n \leq 1. \]

This model is immediately ruled out

It follows that the above results are consistent with ones in

\textit{PRD76}(2013), gr-qc/1308.3401v1
The energy conditions (SEC, NEC, DEC, WEC) and the Dolgov-Kawasaki stability criterion in the $f(R, L_m)$ gravity with arbitrary coupling between matter and geometry have been derived, which can degenerate to the ones in the special cases including the non-minimal coupled and pure $f(R)$ gravities as well as GR;

Constraints on a class of $Gf(R)G$ models have been explicitly given in terms of the above conditions;

The conditions for late-time cosmic accelerated expansion in the $Gf(R)G$ models, which are consistent with ones in \textit{PRD}76(2013), gr-qc/1308.3401v1.
Thank you!