A Tangent Bundle Formulation of Relativistic Kinetic Theory: A Few Applications

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Outline

- Geometry of the Tangent bundle
- Model for a simple collisionless relativistic gas-Liouvilles equation
- On Solutions of Liouvilles equation on a Kerr black hole
- Conclusions

Geometry of the the tangent bundle

Let (M,g) an $n=d+1$-dimension, smooth space-time.

$$TM := \{(x,p) : x \in M, p \in T_xM\} : \quad TangentBundle$$

$\pi : TM \to M : (x,p) \mapsto x : \quad projection \quad map$.

Lemma: *TM is a $2n$-dimension, smooth orientable manifold.* We denote by:

$$T_{(x,p)}(TM) : \quad tangent \quad space \quad at \quad (x,p) \in TM$$

$T_{(x,p)}(TM)$ splits canonicaly into a vertical $V_{(x,p)}$ and a horizontal $H_{(x,p)}$ subspaces:

$$T_{(x,p)}(TM) = H_{(x,p)} \oplus V_{(x,p)}, \quad Z \in T_{(x,p)}(TM) \Leftrightarrow Z = Z^H + Z^V.$$

Here after (x^μ, p^μ), $\mu \in \{1, 2, ...n\}$ local adopted coordinates on TM

Geometry of the the tangent bundle

- $\pi_{*(x,p)} : T_{(x,p)}(TM) \to T_xM$: push-forward

In adapted local coordinates (x^μ, p^μ) we have, for $Z \in T_{(x,p)}(TM)$,

$$Z = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{(x,p)} + P^\mu \left.\frac{\partial}{\partial p^\mu}\right|_{(x,p)}, \qquad \pi_{*(x,p)}(Z) = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_x .$$

$$V_{(x,p)} := \ker \pi_{*(x,p)} = \{Z \in T_{(x,p)}(TM) : \pi_{*(x,p)}(Z) = 0\}$$

The Connection map $K_{(x,p)} : T_{(x,p)}(TM) \to T_xM : Z \to K_{(x,p)}(Z)$

$$K_{(x,p)}(Z) = K_{(x,p)}(X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{(x,p)} + P^\mu \left.\frac{\partial}{\partial p^\mu}\right|_{(x,p)}) = \left[P^\mu + \Gamma^\mu{}_{\alpha\beta}(x) X^\alpha p^\beta\right] \left.\frac{\partial}{\partial x^\mu}\right|_x .$$

$$H_{(x,p)} := \ker K_{(x,p)} = \{Z \in T_{(x,p)}(TM) : K_{(x,p)}(Z) = 0\}.$$

$H_{(x,p)}$ is spanned by: $e_\mu := \left.\frac{\partial}{\partial x^\mu}\right|_{(x,p)} - \Gamma^\alpha{}_{\mu\beta} p^\beta \left.\frac{\partial}{\partial p^\alpha}\right|_{(x,p)}, \quad \mu \in \{1, 2, ...n\}$

Geometry of the the tangent bundle

We now introduce the Sasaki metric $\hat{g}$ on TM defined by

$$\hat{g}(X,Y) := g(\pi_*(X),\pi_*(Y)) + g(K(X),K(Y)),$$

$$\hat{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu + g_{\mu\nu}\theta^\mu \otimes \theta^\nu, \qquad \theta^\mu = dp^\mu + \Gamma^\mu{}_{\alpha\beta}p^\beta dx^\alpha.$$

- $\hat{g}$ is a Semi-Riemmanian metric of signature $(-,-,+,+,+,+,+,.....,+)$.
- $\hat{g}$ makes the spliting $T_{(x,p)}(TM) = H_{(x,p)} \oplus V_{(x,p)}$ orthogonal.
- $\hat{g}$ defines a natural symplectic form Ω_s on TM

$$\Omega_s(X,Y) := \hat{g}(X,J(Y)).$$

- J is an almost complex structure $J : TM \to TM$ defined by

$$J(Z^H) := Z^V, \quad J(Z^V) := -Z^H.$$

A model for a Simple Rel. Gas

We now use these geometrical structures of the tangent bundle to describe Relativistic kinetic theory of a collisionless simple gas propagating on a connected and time-orientable (M, g) i.e.:

- a collection of spinless, classical particles all of the same rest mass $m > 0$
- particles move along future directed timelike geodesics of the background (M, g)

For the tangent bundle description of this gas we:

- Introduce the Liouville vector field on TM:

$$L := (I^H)^{-1}(p) = p^\mu e_\mu = p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu}.$$

- Introduce the Hamiltonian function on TM:

$$H(x,p) := \frac{1}{2}\hat{g}_{(x,p)}(L, L) = \frac{1}{2} g_x(p,p) = \frac{1}{2} g_{\mu\nu}(x) p^\mu p^\nu.$$

A model for a Simple Rel. Gas

- Define the mass shell

$$\Gamma_m := H^{-1}\left(-\frac{m^2}{2}\right) = \{(x,p) \in TM : g_x(p,p) = -m^2\}.$$

- Γ_m is a $(2n-1)$-dim. Lorentzian submanifold of TM.
- For (M,g) connected and time-orientable, then $\Gamma_m = \Gamma_m^+ \cup \Gamma_m^-$ i.e. Γ_m is the disjoin union of the future Γ_m^+ and past mass shell Γ_m^-

In the following we assume (M,g) to be time-oriented and work on the future mass shell Γ_m^+ (gas particles move towards the future).

A model for a Simple Rel. Gas

For the statistical description of the gas we introduce the distribution function $f : \Gamma_m^+ \to \mathbb{R}$ and the current density

$$\mathcal{J} := fL/m.$$

Physical interpretation: Let Σ be a $(2n-2)$-dimensional spacelike hypersurface in Γ_m with normal vector field ν, then the flux integral

$$N[\Sigma] = -\int_{\Sigma} \hat{g}(\mathcal{J}, \nu) d\Sigma$$

is the averaged number of occupied trajectories that intersect Σ.

For a collisionless gas the distribution function f must satisfy:

$$\pounds_L f = p^\mu \frac{\partial f}{\partial x^\mu}(x,p) - \Gamma^\mu{}_{\alpha\beta}(x) p^\alpha p^\beta \frac{\partial f}{\partial p^\mu}(x,p) = 0 \quad \mathit{Liouviles} \quad \mathit{equation}.$$

Applications

As an application, we derive the most general collisionless distribution function on a Kerr black hole background.

Strategy: Find a canonical transformation on TM that trivializes the Liouville vector field: $(x^\mu, p^\mu) \mapsto (Q^\alpha, P^\alpha)$ such that $L = \frac{\partial}{\partial Q_0}$.

This can be achieved by using the Hamilton-Jacobi (HJ) method. Solve the HJ equation

$$H(x, \nabla S) = -\frac{1}{2}m^2 \Leftrightarrow g_x(\nabla S, \nabla S) = -m^2,$$

where $S = S(x, P)$ is the generating function:

$$p_\mu = \frac{\partial S}{\partial x^\mu}, \qquad Q^\alpha = \frac{\partial S}{\partial P_\alpha}.$$

Leaves the symplectic form invariant: $\Omega_s = dp_\mu \wedge dx^\mu = dP_\alpha \wedge dQ^\alpha$.

For the Kerr spacetime the HJ equation is separable (Carter, ’68).

Applications

Complete solution has the form

$$S(t,\varphi,r,\vartheta,m,E,\ell_z,\ell) = -Et + \ell_z\varphi + \int^r \sqrt{R(r)}\frac{dr}{\Delta(r)} + \int^\vartheta \sqrt{\Theta(\vartheta)}d\vartheta,$$

where

$\Delta(r) = r^2 - 2m_H r + a_H^2,$

$R(r) = \left[(r^2+a_H^2)E - a_H\ell_z\right]^2 - \Delta(r)(m^2r^2+\ell^2),$

$\Theta(\vartheta) = \ell^2 - \left(\frac{\ell_z}{\sin\vartheta} - a_H\sin\vartheta E\right)^2 - m^2a_H^2\cos^2\vartheta$

and

- m: rest mass of particles
- $E = -p_t$: conserved energy
- $\ell_z = p_\varphi$: conserved angular momentum
- ℓ^2: Carter constant

Applications

Explicitly, the new coordinates (Q, P) are given by

$$P_0 := m, \qquad P_1 := E, \qquad P_2 := \ell_z, \qquad P_3 := \ell,$$

$$Q_0 := \frac{\partial S}{\partial m} = -m \int^r \frac{r^2 dr}{\sqrt{R(r)}} - m a_H^2 \int^\vartheta \frac{\cos^2 \vartheta d\vartheta}{\sqrt{\Theta(\vartheta)}},$$

$$Q_1 := \frac{\partial S}{\partial E} = -t + \int^r \frac{(r^2 + a_H^2) A(r)}{\sqrt{R(r)}} \frac{dr}{\Delta(r)} + a_H \int^\vartheta \frac{B(\vartheta)}{\sqrt{\Theta(\vartheta)}} d\vartheta,$$

$$Q_2 := \frac{\partial S}{\partial \ell_z} = \varphi - a_H \int^r \frac{A(r)}{\sqrt{R(r)}} \frac{dr}{\Delta(r)} - \int^\vartheta \frac{B(\vartheta)}{\sqrt{\Theta(\vartheta)}} \frac{d\vartheta}{\sin^2 \vartheta},$$

$$Q_3 := \frac{\partial S}{\partial \ell} = -\ell \int^r \frac{dr}{\sqrt{R(r)}} + \ell \int^\vartheta \frac{d\vartheta}{\sqrt{\Theta(\vartheta)}},$$

with the functions $A(r) := (r^2 + a_H^2)E - a_H \ell_z$ and $B(\vartheta) := \ell_z - a_H \sin^2 \vartheta E$.

Applications

By construction, $H = -m^2/2 = -P_0^2/2$ in terms of the new coordinates (Q, P). Therefore,

$$\dot{Q}_0 = \frac{\partial H}{\partial P_0} = -m,$$

while all the other Q's and all the P's are constant.
Consequently, the Liouville vector field in these new coordinates is simply

$$L = -m \frac{\partial}{\partial Q_0}.$$

Therefore, the most general collisionless distribution function on Kerr is

$$f(x, p) = F(Q_1, Q_2, Q_3, P_0, P_1, P_2, P_3).$$

- f is stationary and axisymmetric if F is independent of Q_1 and Q_2.
- Solution is only formal: Q's are multi-valued in general!

Conclusions

- Previous method has been extended to:
- Case of a charged collisionless gas on a Kerr-Newman balck hole
- Case of a collisionless gas propagating on a FRW space times

References

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