The Spacetime Geometry of a Null Electromagnetic Field

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Einstein-Maxwell Equations

g : Electrovacuum Metric

F : Electromagnetic 2-form

$$G(g) = F \cdot F + {*F} \cdot {*F}$$

$$dF = 0 = d{*F}$$

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Jordan and Kundt (1961), Geroch (1966), Ludwig (1970)

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C. G. Torre
arXiv:1308.2323

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For example, in the twist-free case :

$$\Re\left\{(\overline{\delta}+\overline{\beta}-\alpha)(\tau-2\beta)-\frac{1}{2}(\mu-\overline{\mu})(\epsilon-\overline{\epsilon})\right\}=0$$

(Newman-Penrose formalism)

5th-order conditions in the twisting case

$$
\begin{aligned}
&\omega\delta\Big\{\Re\Big[\delta(\overline{\tau}-2\overline{\beta})\Big]\Big\}-\Big[\delta\omega+\omega(\tau-\overline{\alpha}-\beta)\Big]\Big[\Re\left\{(\bar{\delta}+\overline{\beta}-\alpha)(\tau-2\beta)\right\}\\
&+i\Im(\mu)(\rho-2\epsilon)\Big]+\frac{\omega}{2}\Big\{\overline{\beta}\delta(2\overline{\alpha}+\tau-4\beta)+\beta\delta(2\alpha+\overline{\tau}-4\overline{\beta})+2i\delta(\Im(\mu)(\rho-2\epsilon))\\
&+\tau\delta(\overline{\beta}-\alpha)+\overline{\tau}\delta(\beta-\overline{\alpha})-\alpha\delta(\tau-2\beta)-\overline{\alpha}\delta(\overline{\tau}-2\overline{\beta})\Big\}-i\omega^2\Delta(\tau-2\beta)\\
&+\omega^2\Big[\overline{\nu}(\omega+2\Im(\epsilon))-(\tau-2\beta)(2\Im(\gamma)+i\mu)+i\overline{\lambda}(\overline{\tau}-2\overline{\beta})\Big]=0
\end{aligned}
$$

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See:

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