Exact solution of Einstein's angular momentum constraint in the angular-momentum sector of matter on the past light cone:

Exact dragging of inertial axes by cosmic energy currents (Mach's principle)

Christoph Schmid, ETH Zurich

27th Texas Symposium on Relativistic Astrophysics
December 8-13, 2013, Dallas, Texas

Inertial Axes $\Leftrightarrow$ Spin Axes of Gyroscopes

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- Argument in opposite direction: In the **local inertial frame** of a gyroscope: **no gyroscope-precession.**

Super-precise observational fact

- Spin-axes of gyroscopes do not rotate **relative to quasars**, except extremely small dragging effect by Earth rotation: **Lense - Thirring effect**, detected by **Gravity Probe B**.
- Gyros at a few Earth radii: Dragging by Earth **negligible.**

Mach's Principle: **The Question**

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- John A. Wheeler: **"Who gives the marching orders"**
  to gyroscope axes (= inertial axes)?

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  General Relativity: **gravito-magnetism** $\Rightarrow$ Lense-Thirring.

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- Mach did not know, **what average** of cosmological masses and their motions should be taken.

Mach's Principle: **Our Results**

- **Our old results**, published in Phys. Rev. 2006 and 2009:

  exact dragging of inertial axes (Mach's Principle)
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- **Our new results**, unpublished:

  Exact angular momentum constraint equation
  for matter angular-momentum on **past light-cone**:
  E X A C T Einstein equation: L I N E A R ord. diff. eq.

  Mach's postulate $\Rightarrow$ **L I N E A R relation input, output.**

Significance: Is **rotation absolute or relative?**

- **L O C A L analysis** in laboratory, solar system, galaxy:
- **Analysis** in **solar system**: **Newton correct** with
  ABSOLUTE-UNAMBIGOUS non-rotating frame.
- **General Relativity** of **solar system** (e.g. perihelion shift):
  Local non-rotating frames:
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  Asymptotic **nonrotating frame**: **observational input**,
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  ABSOLUTE–NO-CAUSE–input-by-hand
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- **G L O B A L C O S M O L O G I C A L analysis:**
- Local non-rotating system is
  nonrotating RELATIVE to average of cosmic matter:
  Cosmological GR is a "Theory of Relativity".

NON-PERTURBATIVE cosmological geometry

- Spirit of **Riemann normal coord.** on arbitrary geometry.
- **Observation** event P_0: **World-lines of photons** observed give **radial coordinate lines** on past light-cone (LC).
- **Observer at rest relative to asymptotic quasars** $\Leftrightarrow$ **no dipole term** in measured **Hubble velocities**.

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$$P \quad \Rightarrow \quad \textbf{coordinate} \;\; r_P \quad \text{with} \quad g_{rr} = 1.$$

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- Choose **polar quasar** and **zero-meridian quasar**

$$P \quad \Rightarrow \quad \textbf{coordinates} \;\; (\theta_P, \phi_P).$$

- **Equilateral triangles, infinitely narrow**: two **neighboring** photon world-lines with equal spatial distances $r_{OP} = r_{OP'}$ generate **2-spheres orthogonal** to photon world-lines:

$$g_{r\theta} = 0, \qquad g_{r\phi} = 0.$$

Gyroscope precession
$\Rightarrow$ toroidal vorticity sector for cosmic matter flow

- **Non-perturbative** treatment. **Exact fields.**
- On **one** past light-cone: decomposition in **3-scalar, 3-vector, 3-tensor sectors** rigorously valid.
- On **one** past light-cone: We **neglect tensor sector.**
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  vector sector (divergence-free) $\equiv$ **vorticity sector**
  with $\boldsymbol{J^P = 1^+}$ $\Rightarrow$ **toroidal vorticity, relative to** $\boldsymbol{P_O}$.
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- **Vector spher. harmonics**, toroidal sector:

$$(V^{(P=+)}_{\ell=1,\, m=0})^{\phi} = \text{constant}, \qquad (V^{(P=+)}_{\ell=1,\, m=0})^{\theta} = 0,$$

give **angular velocity** of **rigid rotation around z-axis.**

Light cones: Intrinsic geometry UNPERTURBED by strong toroidal vorticity

- **Measure circumference C_P** of 2-sphere through P.
  For **simplicity** in this talk, **assume** that $(C_P/r_P) = 2\pi \Rightarrow$ **measured light-cone** indistinguishable from **light cones in Minkowski** and **in spatially flat FRW.**

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- **Tensor spherical harmonics** with **toroidal vorticity**,

$$(T^{(P=+)}_{\ell=1,\, m=0})_{\alpha\beta} \quad = \quad 0.$$

  Metric of 2-spheres must be unperturbed metric,

$$g_{\theta\phi} = 0, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2 \sin^2\theta.$$

- **INTRINSIC 3-GEOMETRY** of **every light-cone (LC)** with **ARBITRARY VORTICITY fields**:
  $\Rightarrow$ **EQUAL** to **UNPERTURBED intrinsic geometry** of **LC** in **Minkowski** and **LC** in **spatially flat FRW.**

From one light cone to a neighbouring one: SHIFT

- **Lapse** function (elapsed measured time between slices at fixed χ) is **unperturbed**, because it is a 3-scalar.
- **Only quantity** referring to toroidal vorticity: **shift-3-vector**

$$\ell = 1,\ m = 0: \quad \beta^{\phi} \equiv \beta(v,\chi), \quad \beta^{\theta} = 0, \quad \beta^{\chi} = 0.$$

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- Hubble $H(r)$ measured

$$\text{comoving coordinate difference} \equiv d\chi \equiv a^{-1}\, dr,$$

at fixed comoving distance: measured time diff. $\equiv dt$,

$$\text{conformal time difference} \equiv d\eta \equiv a^{-1} dt.$$

- **Light-cone coordinates (v, χ, θ, ϕ)** and **metric**:

$$v \equiv \eta + \chi \quad \Rightarrow \quad v \text{ labels past light-cone},$$
$$ds^2 = a^2[-dv^2 + 2dv\, d\chi + \chi^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2\,\beta_{\phi}\,\chi^2 \sin^2\theta\, d\phi\, dv].$$

Exact angular momentum constraint for toroidal vorticity on past light-cone of observation

- **Exact Einstein tensor** for vorticity with $(\ell = 1,\ m = 0)$,

$$2a^4 G^{v\phi}_{(1)} =$$
$$= \partial^2_\chi \beta + (\partial_\chi \beta)\,(4\chi^{-1} - 2\mathcal{H}) + \beta\,(4\mathcal{H}' - 4\mathcal{H}^2),$$

where $\mathcal{H} \equiv a^{-1}(da/d\eta)$ and $\mathcal{H}' \equiv (d\mathcal{H}/d\eta)$.

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- **Matter evolution**: If p/ρ approximated time-independent, $a_0 \equiv 1,\ \eta_0 \equiv 1:\qquad a = \eta^P \quad \Rightarrow \quad \mathcal{H} = P\,\eta^{-1}.$
  Equation for Green function away from δ-function shell,

$$\partial^2_\eta \beta - (\partial_\eta \beta)\,[4(1-\eta)^{-1} - 2P\eta^{-1}]$$
$$- 4\,\beta\,(P + P^2)\,\eta^{-2} = 0,$$

  from $\eta = 0$ (big bang) to $\eta = 1$ (observation): Singular points of regular type, solutions with algebraic singularities. **Numerical solution** of linear diff. eq. of 2nd order needed.

General Relativity $\Rightarrow$ Gravitomagnetism

- Work on **Uniform-Radial-Hubble-Expansion Slices**
- **Operational definitions** via measurements by **Fiducial Observers (FIDOs)** at rest on slice with **L**ocal **O**rtho-**N**ormal **B**ases (**LONBs)** denoted by **hats.**
- Op. Def. of **gravitoelectric field** $\vec{\boldsymbol{E}}_{\mathrm{g}} \equiv \vec{\boldsymbol{g}}$ **:**

$$\frac{d}{dt} p_{\hat{i}} \equiv m\, E^{\mathrm{g}}_{\hat{i}} \quad \text{free-falling } \textbf{quasistatic} \text{ test particle,}$$

- Op. Def. of **gravitomagnetic field** $\vec{\boldsymbol{B}}_{\mathrm{g}}$ **:**

$$\Omega^{\mathrm{gyro}}_{\hat{i}} \equiv -\frac{1}{2} B^{\mathrm{g}}_{\hat{i}} \quad \text{precession of gyro comoving with FIDO.}$$

- **Gravitomagnetic vector potential** $\vec{\boldsymbol{A}}_{\mathrm{g}}$ **:** $\vec{B}_{\mathrm{g}} = \operatorname{curl} \vec{A}_{\mathrm{g}}$
  it follows : $\vec{\boldsymbol{A}}_{\mathrm{g}} = \vec{\beta} \equiv$ **shift vector.**

Einstein's $G^{\hat{o}}_{\hat{k}}$ Eq.: **Angular Momentum Constraint**

- **NEW:** Angular momentum constraints for **Uniform-Radial-Expansion Slices** are **form-identical** for slices which are spatially flat, hyperbolic, or spherical,

$$\boldsymbol{(-\Delta + \mu^2)\, \vec{A}_{\rm g} = -16\pi G_{\rm N}\, \vec{J}_{\varepsilon},}$$

  $(\mu/2)^2 \equiv -(dH/dt) \equiv (H\text{-dot radius})^{-2}$,
  $\vec{J}_{\varepsilon} =$ **energy current density** = **momentum density**

- **NEW:** Same as Ampère's law for stationary magnetism except $\mu^2\,\vec{A}_{\rm g} \Rightarrow$ **Yukawa suppression (*H*-dot radius). Elliptic equation** (no partial time-derivatives of perturbations) **for time-dependent gravitomagnetism.**

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- **NEW:** $\vec{J}^{\varepsilon}_{\hat{k}} = T^{\hat{0}}_{\hat{k}}$ is **measurable input** without prior knowledge of g_{0i}, which is output. $T^0_{\ k}$ **not** measurable.
- **NEW** in cosmology: **Laplacian** on Vector Fields in Riemannian 3-Spaces is **de Rham - Hodge Laplacian.**

Solution of the **Angular Momentum Constraint**

- **Identical expressions** for solutions of cosmological gravitomagnetism on **Uniform-Radial-Expansion slices**, which are **spatially open**,

$$\vec{B}_g(P) = -2\,\vec{\Omega}_{\text{gyro}}(P) =$$
$$= -4\,G_N \int d(\text{vol}_Q)\,[\vec{n}_{PQ} \times \vec{J}_\varepsilon(Q)]\,Y_\mu(r_{PQ})$$
$$Y_\mu(r) = \frac{-d}{dr}\,[\frac{1}{R}\exp(-\mu r)] = \text{Yukawa force}$$

$r =$ radial distance, $2\pi R =$ circumference of great circle.

- **NEW:** Solution **form-invariant if one goes to rotating frame** $\Leftrightarrow$ **Only relative motion** of quasars and gyroscope spins is **relevant.**

Bottom Line: Exact Dragging of Inertial Axes

Mach's Principle holds for Vorticity, strong or weak

- Inertial axes **exactly follow** the **weighted average of cosmic energy currents** (in all reference frames):

$$\vec{\Omega}_{\rm gyro} = < \vec{\Omega}_{\rm matter} > \equiv \int_0^\infty dr \; \vec{\Omega}_{\rm matter}(r) \; W(r),$$

$$W(r) = \frac{1}{3} 16\pi G_{\rm N}(\rho + p) R^3 \; Y_\mu(r).$$

  This is the **exact solution** for vorticity on **Uniform-Radial-Expansion Slices** which are **spatially open.**

- Resulting weight function $W(r)$ in **exact solution normalized** to **unity** $\Leftrightarrow$ **exact rotational frame-dragging.**