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THE Conformal Stealth on Cosmology

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Is common practice in GR that any component of the matter content will necessarily leave a trace in the spacetime geometry.

Nevertheless, there are special nontrivial matter configurations with no backreaction on the gravitational field.

Scalar fields with this property have been found for the static BTZ black hole[1], Minkowski flat space [2], and (A)dS space[3].

They were coined gravitational *stealths*.

For the de Sitter case, there is a cosmological version[4] and an aplication[5].

[1] E.A-B, C.M., J.Z. hep-th/0403228
[2] E.A-B, C.M., R.T., J.Z hep-th/0505086
[3] E.A-B, C.M., R.T., J.Z. (in preparation)
[4] N.B., R.K.J., D.P.J. hep-th/0610109
[5] H.M., K.I.M. gr-qc/1208.5777

- The equations determining the stealth can be interpreted as demanding the gravitational background to be an extremal of the related stealth action only.
- Under this interpretation, a stealth configuration (background and SF), can be linked to another potentially different configuration via any symmetry transformation of the action.
- In particular, having a concrete example of stealth for a conformally invariant action implies that its whole conformal class allows also a stealth.
- Since FRW, have vanishing Weyl tensor, this implies that FRW is conformally flat: obvious for $k = 0$, but less trivial for curved universes, $k = \pm 1$.

BTZ BH.

BTZ BH is a solution with zero Cotton tensor $\Longrightarrow$ conformally flat $\Longrightarrow$ locally diffeomorphic to AdS_3.

And since AdS is conformally flat in any dimension, this property can be used to find a stealth on AdS_D starting on Minkoski.

However, by using the diffeomorphism between BTZ and AdS_3 to find the corresponding stealth on BTZ, after making the identification $\phi \to \phi + 2\pi$, it is found that the expresion is multi-valued. Imposing to be univalued set $J = 0$.

So, there is no stealth on rotating BTZ BH, despite there exists the local conformal transformation.

The cosmological principle leads to a space time with the FRW metric

$$ds^2 = a(\tau)^2 \left(-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right),$$

which for a given matter content φ_{m} extremize the action

$$S[g, \varphi_{\mathrm{m}}] = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa}(R - 2\Lambda) + L_{\mathrm{m}} \right),$$

by solving the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa T_{\mu\nu} = 0.$$

we next introduce the action

$$S[g,\varphi_{\mathrm{m}}]-\frac{1}{2}\int d^4x\sqrt{-g}\left(\partial_\mu\Psi\partial^\mu\Psi+\frac{1}{6}R\Psi^2+\lambda\Psi^4\right).$$

Our aim is to find those critical configurations with the special property that both sides of Einstein equations vanish independently

$$0=G_{\mu\nu}+\Lambda g_{\mu\nu}-\kappa T_{\mu\nu}=\kappa T^{\mathrm{s}}_{\mu\nu}=0,$$

where

$$\begin{aligned} T^{\mathrm{s}}_{\mu\nu} &= \partial_\mu\Psi\partial_\nu\Psi-\frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\Psi\partial^\alpha\Psi+U(\Psi)\right) \\ &\quad+\frac{1}{6}\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+G_{\mu\nu}\right)\Psi^2. \end{aligned}$$

In order to prove the existence of a stealth in any standard cosmology we need to show that imposing the vanishing of the energy-momentum tensor $T^{\mathrm{s}}_{\mu\nu}$ evaluated in the FRW background is compatible with a nontrivial scalar behavior: the stealth.

We find useful to define te auxiliar function

$$\Psi = \frac{1}{a\sigma},$$

where the function $\sigma = \sigma(x^{\mu})$ inheritates the full spacetime dependence of the scalar field.

By solving the off diagonal elements of $T^{s}{}_{\mu\nu}$ we find that

$$\sigma(x^{\mu}) = T(\tau) + R(r) + r\left[\Theta(\theta) + \sin(\theta)\Phi(\phi)\right].$$

note that, there exists a freedom in the election of the above functions, concretely, homogeneous terms in Φ, Θ, and R can be compensated by a sinusoidal dependence in Θ, a linear one in R, and another homogeneous term in T, respectively.

This is called residual symmetry.

Working with combinations of the diagonal components we arrive to

$$\begin{aligned}
\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}+\Theta &= (1-kr^2)r\frac{\mathrm{d}^2R}{\mathrm{d}r^2}-\frac{\mathrm{d}R}{\mathrm{d}r},\\
\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}+\Phi &= \sin\theta\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}-\cos\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta},
\end{aligned}$$

whose solutions are

$$\begin{aligned}
\Phi(\phi) &= A_1\cos\phi+A_2\sin\phi,\\
\Theta(\theta) &= A_3\cos\theta,\\
R(r) &= \begin{cases} B_-\sqrt{1-kr^2}, & k\neq 0,\\ \frac{1}{2}\alpha r^2, & k=0, \end{cases}
\end{aligned}$$

Finally we use the combination $3a^4\sigma^3\left(T^{\mathrm{s}}{}_{\theta}{}^{\theta}-T^{\mathrm{s}}{}_{t}{}^{t}\right)$ to obtain

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}T+kT=0, \quad k\neq 0,\\ &\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}T+\alpha=0, \qquad k=0, \end{aligned},$$

Of course the solution is

$$T(\tau)=\begin{cases} -\frac{A_0}{\sqrt{k}}\sin\left(\sqrt{k}\tau\right)+B_{+}\cos\left(\sqrt{k}\tau\right), & k\neq 0,\\ -\frac{1}{2}\alpha\tau^2-A_0\tau+\sigma_0, & k=0, \end{cases}$$

Only one equation remains in our study, and allows to relate the coupling constant of the potential with the integration constants. Let be $B_{\pm} = \sigma_0/2 \pm \alpha/k$,

$$-2a^4\sigma^4 T^{\mathrm{s}}{}_t{}^t = \lambda + {A_0}^2 - \vec{A}^2 + 2\alpha\sigma_0 = 0,$$

One of the integration constants is fixed in terms of the coupling constant of the conformal potential and the other integration constants.

For universes with flat spatial topology, $k = 0$, the FRW metric is manifestly (global) conformally flat.

In this case we indeed expect the expression for the stealth found to be exactly a conformal.

$$\Psi = \frac{1}{a}\Psi_{\text{flat}}.$$

which is achieved by re-writing

$$\sigma(x^\mu) = \frac{\alpha}{2}x_\mu x^\mu + A_\mu x^\mu + \sigma_0,$$

This is precisely the result for flat spacetime

The above results are easy to generalize to any number of dimensions. The conformal stealth must be written now as

$$\Psi = \frac{1}{(a\sigma)^{(D-2)/2}} .$$

Then, in the D-dimensional version of action, the conformal coupling must be generalized to

$$\frac{1}{6} \quad \longrightarrow \quad \frac{D-2}{4(D-1)},$$

and the conformal potential takes the form

$$\frac{1}{2}\lambda\Psi^4 \quad \longrightarrow \quad \frac{(D-2)^2}{8}\lambda\Psi^{2D/(D-2)}.$$

For the curved case

$$ds^2 = a(\tau)^2 \left(-d\tau^2 + d\vec{x}^2 + \frac{k\,(\vec{x}\cdot d\vec{x})^2}{1 - k\,\vec{x}^2} \right).$$

and the stealth is

$$\begin{aligned} \sigma(x^\mu) &= -\frac{A_0}{\sqrt{k}} \sin\left(\sqrt{k}\tau\right) + B_+ \cos\left(\sqrt{k}\tau\right) \\ &\quad + \vec{A}\cdot\vec{x} + B_- \sqrt{1 - k\,\vec{x}^2}, \end{aligned}$$

being $B_\pm = \frac{\sigma_0}{2} \pm \frac{\alpha}{k}$. As a byproduct these redefinitions allows to recover consistently the flat case by taking the limit $k \to 0$

The generalization to higher dimensions is obvious from the former expresions

Because of the curvature, there is not invariance under translations, however a generalization is kept as symmetry The **quasitraslations**:

$$\vec{x} \mapsto \vec{x} + \vec{a} \left(\sqrt{1 - k\,\vec{x}^2} - \frac{\left(1 - \sqrt{1 - k\,\vec{a}^2}\right)}{\vec{a}^2} \vec{a}\cdot\vec{x} \right) .$$

(If $k = 0$ is a normal translation). It is a transformation that sends the origin $\vec{x} = \vec{0}$ to $\vec{x} = \vec{a}$, which is a realization of the homogeneous character of FRW space.

Under quasitraslations, the metric is invariant but the stealth is only form invariant, that means that it have transformed constants also. This allows to choose specific parameters $\vec{a}$ in order to have vanishing $\vec{A}$, which is achieved by selecting

$$\vec{a} = \frac{\vec{A}}{\sqrt{\left(\alpha - \frac{k}{2}\sigma_0\right)^2 + k\vec{A}^2}}.$$

Because of that, the curved stealth have only 2 integration constants

Consider[6]

$$
\begin{aligned}
\tau &= \frac{1}{\sqrt{k}} \arctan\left(\frac{\sqrt{k}t}{1-\frac{k}{4}(t^2-\rho^2)}\right), \\
r &= \frac{1}{\sqrt{k}} \sin\left(\arctan\left(\frac{\sqrt{k}\rho}{1+\frac{k}{4}(t^2-\rho^2)}\right)\right),
\end{aligned}
$$

[6] A. Lightman et al, *Problem book in Relativity and Gravitation* (Princeton, 1975).

with inverse given by

$$t = \frac{2\sin(\sqrt{k}\tau)}{\sqrt{k}\left(\cos(\sqrt{k}\tau)+\sqrt{1-kr^2}\right)},$$

$$\rho = \frac{2r}{\cos(\sqrt{k}\tau)+\sqrt{1-kr^2}}.$$

Notice that these are only local transformations.
If $k = 1$ the whole MST maps in a patch of FRW, and viceversa

Then, in D-dimensions we can write

$$
\begin{aligned}
ds^2_{\mathrm{FRW}} &= a(\tau)\left(-d\tau^2 + \frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{D-2}\right) \\
&= \frac{a^2(\tau(t,\rho))\left(-dt^2 + d\rho^2 + \rho^2 d\Omega^2_{D-2}\right)}{\left(\frac{k}{4}(\rho+t)^2+1\right)\left(\frac{k}{4}(\rho-t)^2+1\right)}.
\end{aligned}
$$

which means that the stealth transfomation would be

$$
\begin{aligned}
\Psi_{\mathrm{FRW}} &= \frac{1}{\Omega^{(D-2)/2}}\Psi_{\mathrm{M}} \\
&= \frac{1}{(\Omega\sigma_{\mathrm{M}})^{(D-2)/2}},
\end{aligned}
$$

and now the auxiliar function can be re-written like

$$
\sigma_{\mathrm{M}} = \left[\frac{\alpha}{2}(-t^2+\rho^2) - A_0 t + \rho A_m \pi^m + \sigma_0\right],
$$

where π^m are the polar coordinates of the unit sphere S^{D-2}.

Using the inverse transformations, the conformal factor reduces to

$$\Omega = \frac{a(\tau)}{2}\left[\cos(\sqrt{k}\tau) + \sqrt{1-kr^2}\right],$$

while for σ_{M} we obtain

$$\begin{aligned}\sigma_{\mathrm{M}} = \quad & 2\Bigg(- \frac{\alpha}{k}\frac{\cos(\sqrt{k}\tau) - \sqrt{1-kr^2}}{\cos(\sqrt{k}\tau) + \sqrt{1-kr^2}} \\ & - \frac{A_0}{\sqrt{k}}\frac{\sin(\sqrt{k}\tau)}{\cos(\sqrt{k}\tau) + \sqrt{1-kr^2}} \\ & \quad + \frac{rA_m\pi^m}{\cos(\sqrt{k}\tau) + \sqrt{1-kr^2}} + \frac{\sigma_0}{2}\Bigg).\end{aligned}$$

some algebra lead us to

$$a\sigma_{\mathrm{FRW}} = \Omega\sigma_{\mathrm{M}}.$$

Conclusions

- We have proved that any homogeneous and isotropic universe, independently of its spatial topology and matter content, allows for the existence of a conformal stealth.
- Sorprisingly, despite the stealth is isotropic, is inhomogeneous.
- Nevertheless, its presence leaves no trace in the cosmological evolution of the given universe.
- Additionally, we have shown that these results are not exclusive of our four-dimensional universe, but are also valid for higher-dimensional generalizations of the FRW spacetime.

- We also discuss to some extension the local conformal arguments that alternatively allow to build such configurations from the ones of Minkowski spacetime.

- After making the explicit construction we show that the potential problems that are known to occur in other contexts, due to the local nature of these arguments, are not present in the cosmological framework.

Perspectives

- Last, but not least, it is important to understand theobservational consequences of the existence of cosmological stealths. In this sense, we note that its fluctuations are not expected to be stealth themselves. This way, the corresponding stealth perturbations may have an imprint on the spectra of the CMB as well as in the statistics of the cosmological large scale structures. Exploring these consequences is the subject of our current research program.

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Acoplamiento mínimo

Hasta ahora no se ha hecho consideración alguna sobre la dimensión o el fondo utilizado.

Considerando el caso MC, $\xi = 0$, el tensor de energía-momentum toma la forma

$$T_{ab} = e_a(\Psi)e_b(\Psi) - \eta_{ab}\left(\frac{1}{2}\nabla_c\Psi\nabla^c\Psi + U(\Psi)\right)$$

Consideremos los elementos sobre la diagonal

$$\begin{aligned} T_{00} &= (e_0(\Psi))^2 - L, \\ T_{ii} &= (e_i(\Psi))^2 + L. \end{aligned}$$

Tomando la suma de cualquier componente espacial con la temporal obtenemos

$$T_{00}+T_{ii}=(e_0(\Psi))^2+(e_i(\Psi))^2=0.$$

Podemos deducir que

$$e_a(\Psi)\equiv 0;$$

así pues Ψ sólo puede ser trivialmente constante; tenemos entonces que $T_{ab}=\eta_{ab}U(\Psi)$.

Concluímos que además Ψ es tal que anula el potencial.

Acoplamiento mínimo

En el caso de acoplamiento mínimo no es posible tener un *stealth*.

Transformaciones conformes

Dada M variedad D-dimensional con métrica g_{ab}: si Ω es una función positiva y suave, entonces decimos que $\tilde{g_{ab}}$

Transformación conforme

$$\tilde{g}_{ab} = \Omega^2 g_{ab}.$$

Las transformaciones conformes aparacen en el contexto de RG en la definición de planicidad asintótica.

Decimos que una ecuación para un campo Ψ es *conformalmente invariante* si existe s número real positivo tal que Ψ es una solución con la métrica g_{ab} si y sólo si

$$\tilde{\Psi} = \Omega^s \Psi$$

es una solución con la métrica $\tilde{g}_{ab}$. Se puede probar que $s = -(D-2)/2$