# Observations in Inhomogeneous Models and the Szekeres Metric

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# Purpose & Plan

- Lay out a general framework for calculating cosmological observables
- Integrate down the PNC of arbitrarily placed observer, in given arbitrary inhomogeneous spacetime
- Propagate observer's coordinates by Lie dragging
- Convert null geodesic eq & geodesic deviation eq to numerical form
- Calculate redshift, diameter distance, proper motions, image distortion
- Apply to Szekeres Metric and test.
- Explore various Szekeres models and their observational patterns.
- Problem of following moving sources
- How to find a later light ray that connects moving source and observer?
- A shooting problem - numerical trial & error?
- We show how to use geodesic deviation equation for instantaneous rates of change.

# PNC & Observations in a General Model

- Calculating observations involves tracing light rays numerically.
- Possible sources observations:
  - ⋆ redshift
  - ⋆ proper motions (flow)
  - ⋆ luminosity/diameter distance
  - ⋆ image distortions (magnification, shear, twist)
  - ⋆ number density
- Initial time — send out a set of rays;
  sources move;
  which directions to send rays at later times?
  seems like big numerical trial & error exercise.
- Use geodesic deviation eq to get rates of change at initial time; & each successive time.

# Observer's Past Null Basis — Setup

- Observer uses angle on sky + time of observation
- Set up observer's coordinates in general inhomogeneous model
  Metric coordinates;
  Orthonormal basis (near observer);
  spherical basis (near observer);
  past-null spherical basis;
  propagate down PNC.

# Observer's Past Null Basis — Define

- Metric & coords (general): $g_{ab}$ , $x^c$ ,
- Observer position (arbitrary): $x^c|_o$ ,
- Orthonormal basis at obs: $\overline{\mathbf{e}}_i\Big|_o = \Big[\overline{e}_i{}^a\,\boldsymbol{\partial}_a\Big]_o$ ,
- Spherical basis at obs:

$$\begin{aligned}
\tilde{\mathbf{e}}_{\tilde{\tau}} &= \overline{\mathbf{e}}_0 \\
\tilde{\mathbf{e}}_{\tilde{r}} &= \sin\tilde{\vartheta}\cos\tilde{\varphi}\,\overline{\mathbf{e}}_1 + \sin\tilde{\vartheta}\sin\tilde{\varphi}\,\overline{\mathbf{e}}_2 + \cos\tilde{\vartheta}\,\overline{\mathbf{e}}_3 \\
\tilde{\mathbf{e}}_{\tilde{\vartheta}} &= \tilde{r}\cos\tilde{\vartheta}\cos\tilde{\varphi}\,\overline{\mathbf{e}}_1 + \tilde{r}\cos\tilde{\vartheta}\sin\tilde{\varphi}\,\overline{\mathbf{e}}_2 - \tilde{r}\sin\tilde{\vartheta}\,\overline{\mathbf{e}}_3 \\
\tilde{\mathbf{e}}_{\tilde{\varphi}} &= -\tilde{r}\sin\tilde{\vartheta}\sin\tilde{\varphi}\,\overline{\mathbf{e}}_1 + \tilde{r}\sin\tilde{\vartheta}\cos\tilde{\varphi}\,\overline{\mathbf{e}}_2 \ ,
\end{aligned}$$

- Convert to past-null spherical basis at obs:

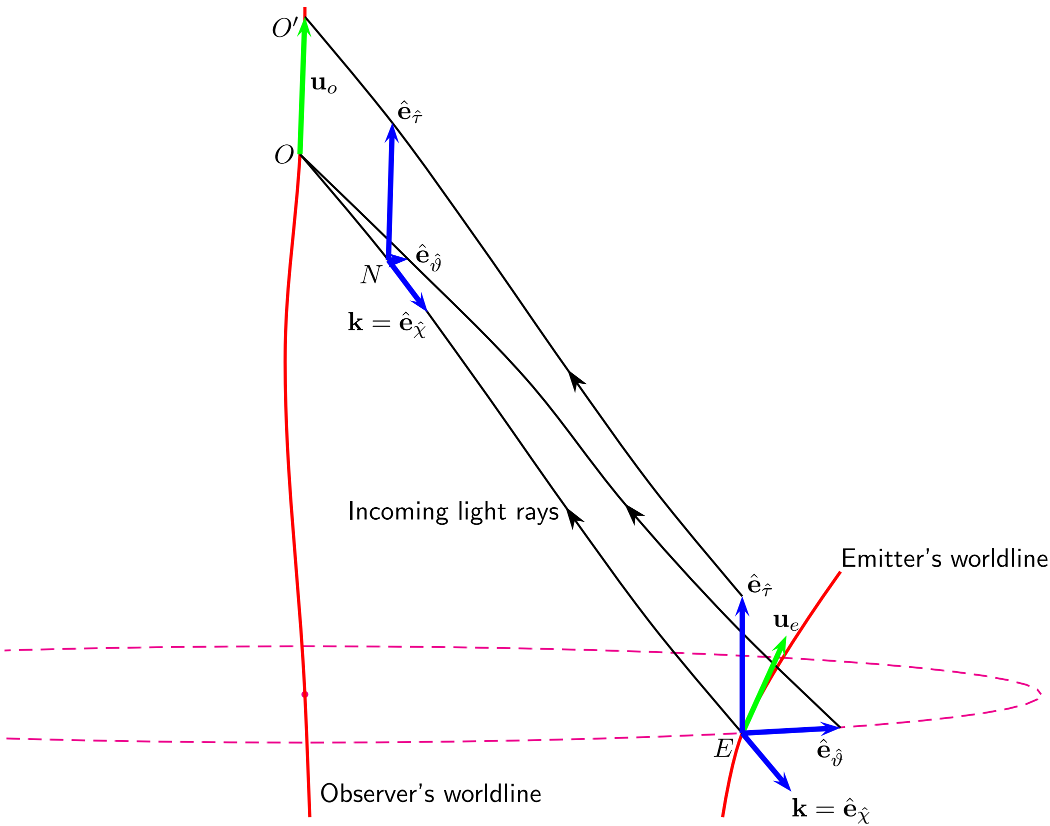
$$\hat{\tau} = \tilde{r} + \tilde{\tau} \ , \quad \hat{\chi} = \tilde{r} \qquad \leftrightarrow \qquad \tilde{\tau} = \hat{\tau} - \hat{\chi} \ , \quad \tilde{r} = \hat{\chi} \ ,$$

$$\begin{aligned} \rightarrow \qquad & \hat{\mathbf{e}}_{\hat{\tau}} = \tilde{\mathbf{e}}_{\tilde{\tau}} = \overline{\mathbf{e}}_0 \\ & \hat{\mathbf{e}}_{\hat{\chi}} = -\tilde{\mathbf{e}}_{\tilde{\tau}} + \tilde{\mathbf{e}}_{\tilde{r}} = -\overline{\mathbf{e}}_0 + \sin\hat{\vartheta}\cos\hat{\varphi}\,\overline{\mathbf{e}}_1 + \sin\hat{\vartheta}\sin\hat{\varphi}\,\overline{\mathbf{e}}_2 + \cos\hat{\vartheta}\,\overline{\mathbf{e}}_3 \\ & \hat{\mathbf{e}}_{\hat{\vartheta}} = \tilde{\mathbf{e}}_{\tilde{\vartheta}} = \hat{\chi}\cos\hat{\vartheta}\cos\hat{\varphi}\,\overline{\mathbf{e}}_1 + \hat{\chi}\cos\hat{\vartheta}\sin\hat{\varphi}\,\overline{\mathbf{e}}_2 - \hat{\chi}\sin\hat{\vartheta}\,\overline{\mathbf{e}}_3 \\ & \hat{\mathbf{e}}_{\hat{\varphi}} = \tilde{\mathbf{e}}_{\tilde{\varphi}} = -\hat{\chi}\sin\hat{\vartheta}\sin\hat{\varphi}\,\overline{\mathbf{e}}_1 + \hat{\chi}\sin\hat{\vartheta}\cos\hat{\varphi}\,\overline{\mathbf{e}}_2 \ , \end{aligned}$$

- Propagate these down the observer's PNC.

# Propagation Scheme

- Keep $\hat{\vartheta}$, $\hat{\varphi}$ const along each light ray
  i.e. Lie drag coords & basis down incoming null geodesics.
- Exactly the set-up for geodesic deviation eq to hold.


$O'$
$\mathbf{u}_o$
$\hat{\mathbf{e}}_{\hat{\tau}}$
$O$
$\hat{\mathbf{e}}_{\hat{\vartheta}}$
$N$
$\mathbf{k} = \hat{\mathbf{e}}_{\hat{\chi}}$
Incoming light rays
Emitter's worldline
$\hat{\mathbf{e}}_{\hat{\tau}}$
$\mathbf{u}_e$
$E$
$\hat{\mathbf{e}}_{\hat{\vartheta}}$
Observer's worldline
$\mathbf{k} = \hat{\mathbf{e}}_{\hat{\chi}}$

# Propagation Equations

- Geodesic Eq — light ray paths

$$\frac{\delta k^a}{\delta \hat{\chi}} = 0 \qquad \text{(good for tensor calcs)}$$

$$\frac{\mathrm{d}k^a}{\mathrm{d}\hat{\chi}} = -k^b \Gamma^a{}_{bc} k^c \; , \qquad k^a k_a = 0 \; , \qquad \frac{\mathrm{d}x^a}{\mathrm{d}\chi} = k^a \qquad \text{(good for numerics)}$$

- Geodesic Deviation Eq — past-null-obs basis propagation

$$\frac{\delta^2 \hat{e}_\alpha{}^a}{\delta \hat{\chi}^2} = -R^a{}_{bcd}\, k^b\, \hat{e}_\alpha{}^c\, k^d \qquad \text{(good for tensor calcs)}$$

$$\frac{\mathrm{d}^2 \hat{e}_\alpha{}^a}{\mathrm{d}\hat{\chi}^2} = -k^b \left( 2\Gamma^a{}_{bc} \frac{\mathrm{d}\hat{e}_\alpha{}^c}{\mathrm{d}\hat{\chi}} + \hat{e}_\alpha{}^c k^d \Gamma^a{}_{db,c} \right) \qquad \text{(good for numerics)}$$

$$\hat{\mathbf{e}}_\alpha \;\; \equiv \;\; \{\; \hat{\mathbf{e}}_\tau \; , \;\; \hat{\mathbf{e}}_{\hat{\chi}} = \mathbf{k} \; , \;\; \hat{\mathbf{e}}_{\hat{\vartheta}} \; , \;\; \hat{\mathbf{e}}_{\hat{\varphi}} \;\}$$

- Propagated $(\hat{\tau}, \hat{\chi}, \hat{\vartheta}, \hat{\varphi})$ is a coord system

- Propagated $\hat{\mathbf{e}}_\alpha$ is coord basis — provide transformation between metric and observer's coordinates,

$$\hat{e}^\alpha{}_c = e_c{}^\alpha = \frac{\partial \hat{x}^\alpha}{\partial x^c} \; , \qquad e^c{}_\alpha = \hat{e}_\alpha{}^c = \frac{\partial x^c}{\partial \hat{x}^\alpha} \; .$$

- What we actually need (later) is not $\hat{e}_\alpha{}^a$ but its inverse $\hat{e}^\alpha{}_a$.

# Propagation — Initial Conditions

Geodesic Eq

- In orthonormal frame, initial $k^\alpha = (-1, 1, 0, 0)$, i.e.

$$|k^b u_b u^a|_o = 1 = |k^a(\delta^c_a + u^c u_a)|_o$$

Geodesic Deviation Eq

- $\hat{\mathbf{e}}_{\hat{\tau}}|_o = \mathbf{u}_o$
- Take $\hat{\chi} \to 0$ limits of $\hat{\mathbf{e}}_\alpha$ near-observer expressions
- Fermi-propagate $\mathbf{k}$ along $\mathbf{u}|_o$

$$\left.\frac{\delta k^a}{\delta \tau}\right|_{\hat{\chi}=0} = \left[u^b_o \nabla_b k^a - k_b a^b_o u^a_o + k_b u^b_o a^a_o\right]_{\hat{\chi}=0} = 0$$

and use $$\left.\frac{\delta k^a}{\delta \hat{\tau}}\right|_{\hat{\chi}=0} = \left.\frac{\delta \hat{e}_{\hat{\tau}}{}^a}{\delta \hat{\chi}}\right|_{\hat{\chi}=0} \quad \to \quad \left.\frac{\mathrm{d}\hat{e}_{\hat{\tau}}{}^a}{\mathrm{d}\hat{\chi}}\right|_o$$

- Take $\hat{\vartheta}$ & $\hat{\varphi}$ derivatives of near-observer $\hat{\mathbf{e}}_{\hat{\chi}} = \mathbf{k}$ and use e.g.

$$\left.\frac{\delta k^a}{\delta \hat{\vartheta}}\right|_{\hat{\chi}=0} = \left.\frac{\delta \hat{e}_{\hat{\vartheta}}{}^a}{\delta \hat{\chi}}\right|_{\hat{\chi}=0} \quad \to \quad \left.\frac{\mathrm{d}\hat{e}_{\hat{\vartheta}}{}^a}{\mathrm{d}\hat{\chi}}\right|_o$$

# Observables - Redshift & Proper Motion

- Rate of change of observed angle with respect to observer time

$$\left.\frac{\mathrm{d}\tilde{x}^m}{\mathrm{d}\tilde{\tau}}\right|_o = \left[\frac{\partial \tilde{x}^m}{\partial \hat{x}^\beta}\frac{\mathrm{d}\hat{x}^\beta}{\mathrm{d}\hat{\tau}}\frac{\mathrm{d}\hat{\tau}}{\mathrm{d}\tilde{\tau}}\right]_o = \left[\hat{e}_\beta{}^m \frac{\mathrm{d}\hat{x}^\beta}{\mathrm{d}\hat{\tau}}\right]_o$$

where $$\left[\frac{\mathrm{d}\hat{x}^\beta}{\mathrm{d}\hat{\tau}}\right]_o = \left[\frac{\mathrm{d}\hat{x}^\beta}{\mathrm{d}\hat{\tau}}\right]_e = \left[\frac{\partial \hat{x}^\beta}{\partial x^a}\frac{\mathrm{d}x^a}{\mathrm{d}\tau_e}\frac{\mathrm{d}\tau_e}{\mathrm{d}\hat{\tau}}\right]_e = \frac{\left[\hat{e}^\beta{}_a u^a\right]_e}{(1+z)}$$

$\tau_e$ = the source proper time,
$\tilde{\tau}_o$ = observer's proper time,
$\hat{\tau}$ = its extension down the PNC.

- Hence

$$1 = \left.\frac{\mathrm{d}\hat{\tau}}{\mathrm{d}\hat{\tau}}\right|_o = \frac{\left[\hat{e}^{\hat{\tau}}{}_a u^a\right]_e}{(1+z)} \qquad \rightarrow \qquad 1+z = \left[\hat{e}^{\hat{\tau}}{}_a u^a\right]_e$$

$$\left.\frac{\mathrm{d}\tilde{\vartheta}}{\mathrm{d}\tilde{\tau}}\right|_o = \frac{\left[\hat{e}^{\hat{\vartheta}}{}_a u^a\right]_e}{(1+z)}$$

$$\left.\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tilde{\tau}}\right|_o = \frac{\left[\hat{e}^{\hat{\varphi}}{}_a u^a\right]_e}{(1+z)} .$$

- Note dual basis vectors $\hat{\mathbf{e}}^{\alpha}$ actually what's needed.

# Observables - Diameter Distance

- Take a small displacement at the emitter, $dx^a$
- Measured angular size is

$$\delta^2 = \mathrm{d}\hat{\vartheta}^2 + \sin^2\hat{\vartheta}\,\mathrm{d}\hat{\varphi}^2$$

$$\delta = \sqrt{\left[\,(\hat{e}^{\hat{\vartheta}}{}_a\,dx^a)^2 + \sin^2\hat{\vartheta}\,(\hat{e}^{\hat{\varphi}}{}_b\,dx^b)^2\,\right]_e}$$

- Physical size — projected orthog to line of sight (2-space $\perp$ $\mathbf{k}$ & $\mathbf{u}_e$)

$$D^a = \left[h^a_b j^b_c \mathrm{d}x^c\right]_e$$

- Diameter distance is

$$d_D = \frac{D}{\delta}\,, \qquad D = \sqrt{D^a D_a}$$

# Observables - Image distortion

- Optical tensors no good — can't integrate tensors
- Magnification, shear, twist, encoded in size of basis vectors
- Still working on this
- Probably need one more vector propagated

# Szekeres Metric

$$ds^2 = -\mathrm{d}t^2 + \frac{\left(R' - \frac{RE'}{E}\right)^2 \mathrm{d}r^2}{\epsilon + f} + \frac{R^2}{E^2}\left(dp^2 + dq^2\right)$$

- $E = E(r, p, q)$
- $\epsilon = +1, 0, -1$
- Evolution function $R(t, r)$ same as for LT.
- 6 arbitrary functions of $r$ ($f$, $M$, $a$, $S$, $P$, $Q$)
- No Killing vectors
- V Interesting, not well explored

# Numerics

- Major work by Tony Walters
- ${\Gamma^a}_{bc}$ and ${\Gamma^a}_{bc,d}$ not small!
- Extensively tested
- Still to be improved

# Testing

- Analytic results for simple RW special case:
$M = M_0 r^3$, $f = -kr^2$, $a = 0$, $\epsilon = +1$, $S = 1$, $P = 0$, $Q = 0$

- Analytic results for simple RW special case

- Full agreement for all scalars, observables, basis vector magnitudes, affine param $\hat{\chi}$, some coord values.

- Not a strong test

- Szekeres-RW special case:
$M = M_0 r^3$, $f = -kr^2$, $a = 0$, $\epsilon = +1$, $S$, $P$, $Q$ not constant
RW in distorted coordinates

- Light paths not constant $p$, $q$.

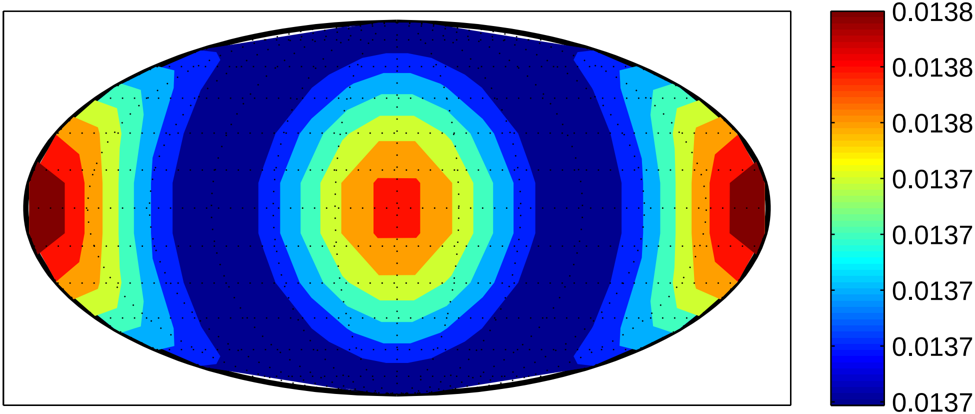
- Still full agreement for all scalars, observables, basis vector magnitudes, affine param $\hat{\chi}$.

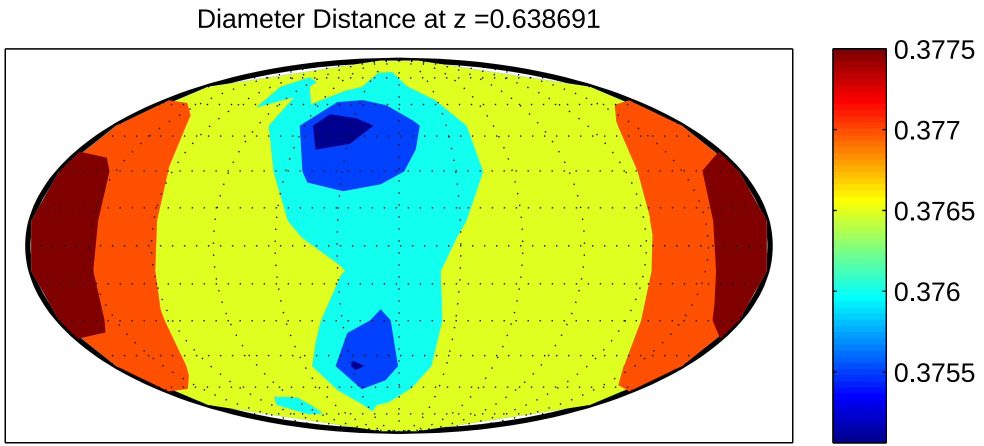
- Significant test

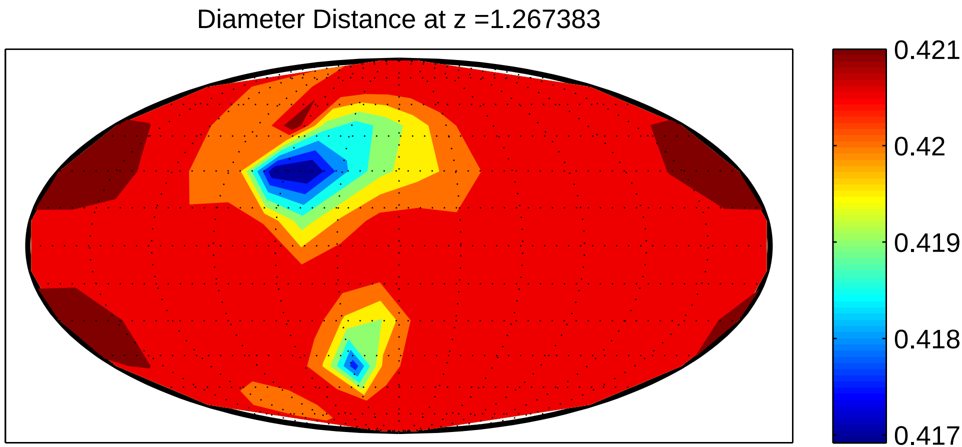
# Results

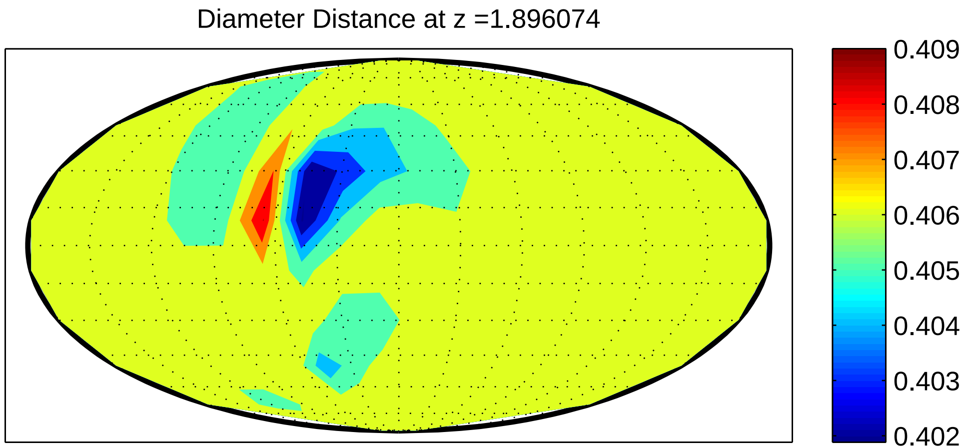
- VERY preliminary
- Not fully checked
- Rather crude:
  Model selection not yet well planned
  Scales not yet properly set
  Points on sky: 10 $\times$ 20 (run time)
- Run 2: $k = -1$, $f = -kr^2$, $M = M_0 r^3 (1 + r)$, $a = 0$,
  $S = 1$, $P = 0$, $Q = r$, $\chi_{end} = 0.54$.
- Following plots show
  set of $d_D$ skymaps at a sequence of $z$ values,
  then set of source proper motions at same $z$ sequence.

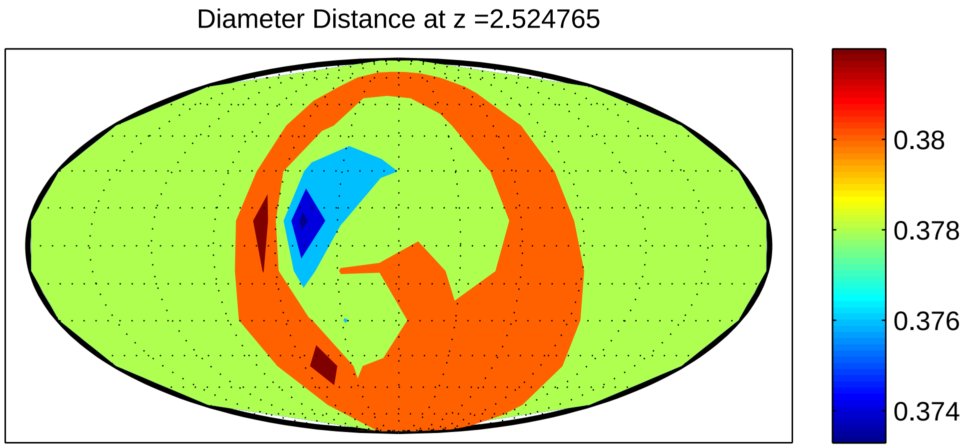
Diameter Distance at z =0.010000



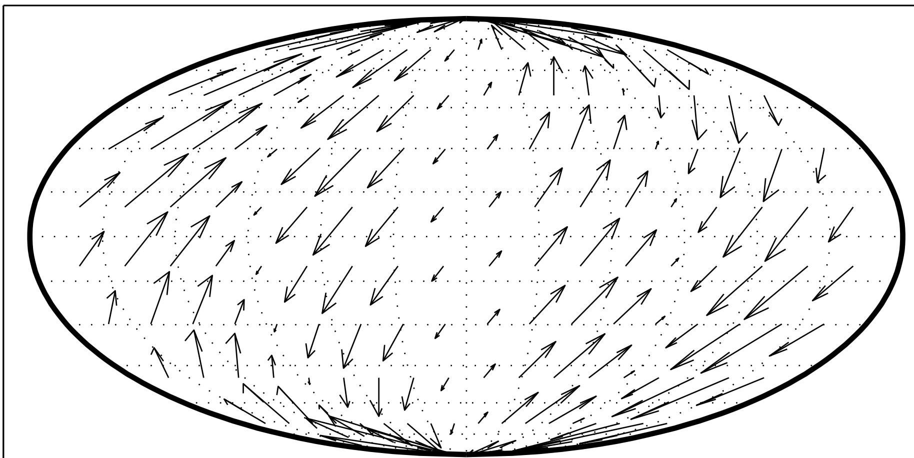

Diameter Distance at z =0.638691
0.3775
0.377
0.3765
0.376
0.3755


Diameter Distance at z =1.267383
0.421
0.42
0.419
0.418
0.417

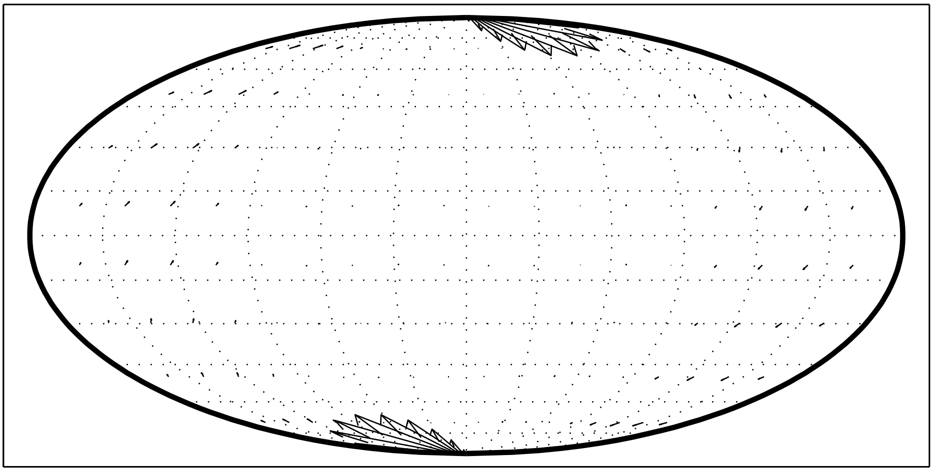

Diameter Distance at z =1.896074
0.409
0.408
0.407
0.406
0.405
0.404
0.403
0.402


Diameter Distance at z =2.524765
0.38
0.378
0.376
0.374

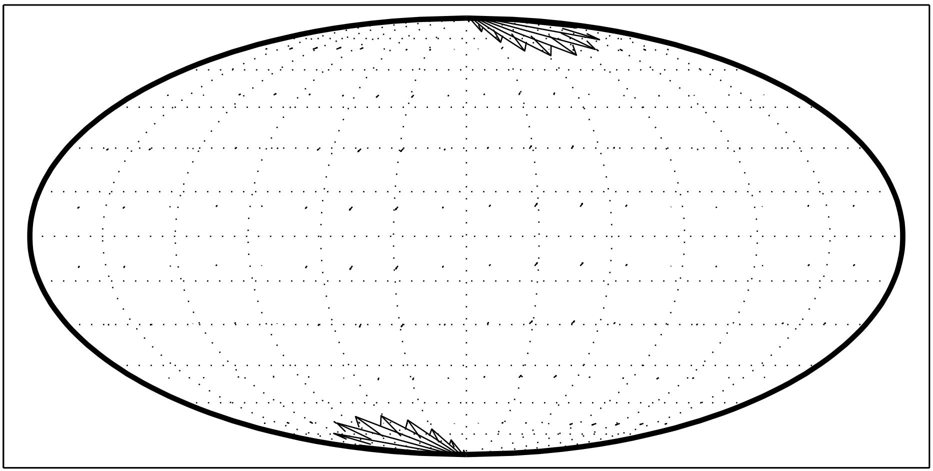
Apparent Motion at z =0.010000



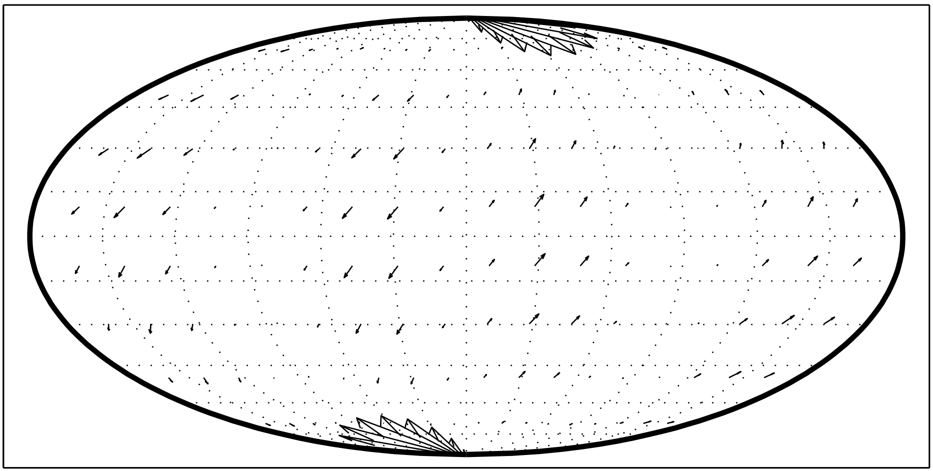
Apparent Motion at z =0.638691



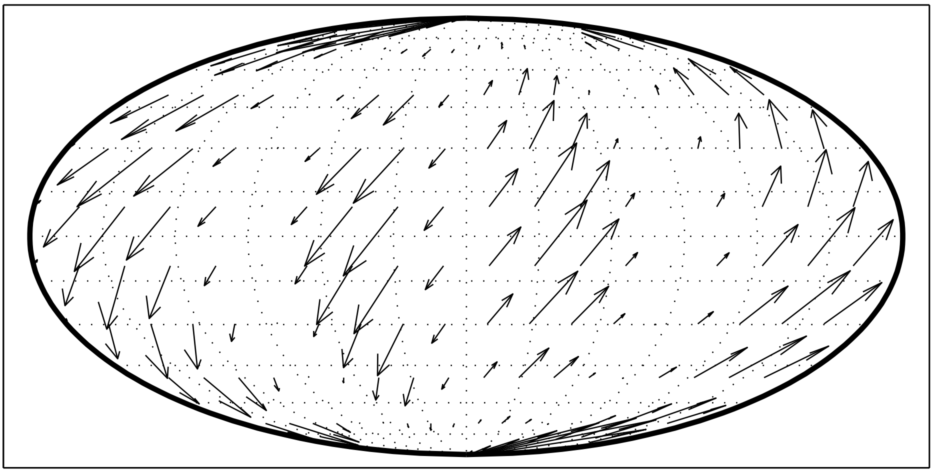
Apparent Motion at z =1.267383



Apparent Motion at z =1.896074



Apparent Motion at z =2.524765



# Discussion

- Method allows calc of observational features for a given observer in a given model
- Method shows source movements, and greatly assists locating correct ray directions to same sources at later times
- Szekeres provides a lot of freedom to tinker, many interesting possibilities
- Important complement to the Metric of the Cosmos Project
- Enables generation of very realistic fake data for testing latter
- Allows checking of results of latter
- Still refining probably a few more little bugs to be removed

**Thankyou!**