Cosmology , the General Theory of Relativity and g- conjugated models

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All the cosmological relativistic models are based on the pseudoriemannian metric g , having Lorentz signature and with the associated Levi-Civita connection (g, ∇). Their diversity and the appearance of the cosmological phenomena require the revaluation of the Einstein model , by adding the new models $M_D = \{(g, D)\}$, with asymmetrical linear connections $\{D\}$. All these models must preserve the ortogonality of the one-dimensional distributions (i.e. of the directions) at the parallel transport (which implies the existence of the nonlinear connection N) and therefore the preservation of the isotropic directions.

From these reasons we will consider the models $M_D = \{(g, D)\}$ with the property that for every linear connections $\overset{(1)}{D}, \overset{(2)}{D} \in M_D$ on M , these linear connections must have the above property. We will say that the linear connections $\overset{(1)}{D}, \overset{(2)}{D}$ are g-conjugated ones (we will denote $\overset{(1)}{D} \overset{g}{\sim} \overset{(2)}{D}$).

Then we will obtain the non-symmetric model of Einstein , as a particular case.

We will also study the relation between N and $\overset{(1)}{D}, \overset{(2)}{D}$, using the modern theory of the vector bundles E and the decomposition $E = HE \oplus VE$ (Whitney). We will give a classification of the g-conjugated cosmological models.

§1. Generalities

A.The differentiable manifolds M, endowed with a pseudo-riemannian metric g, play an essential role in the modelling of a cosmological theory.So we have to study the problem of its existence.

B. The notion of parallel transport is a fundamental one for the geometric modelling. At its turn this notion involve two aspects :

1. The parallel transport which is associated to a linear connection D , on M and , as a particular case , that one which is associated to the Levi-Civitta connection ∇ defined by g. For an arbitrary linear connection D we must study its global existence , on M.

2. The generalised parallel transport which does not take account of a linear connection; if we supplementary take account of a linear connection D we have a particular case.

While the aspect 1. is very often studied the aspect 2. is not so known , but it is very necessary one for a general cosmological theory.

In [5], [13] it was obtained an extended theory of the generalised parallel transport.

Theorem (1.1). Let us consider a paracompact , C^{∞} - differentiable manifold M_n . Then the generalised parallel transport involve a decomposition of the tangent bundle $TM = E$ in the Whitney sum:

(1.1) $TE = HE \oplus VE$

where VE is the vertical subbundle and este HE is the horizontal subbundle

In the mathematical terms we can restate the Theorem 1.1 :

Theorem (1.2). The generalised parallel transport involve the existence of a distribution

(1.2) $H : x \in E \to H_x E \qquad (H_x E \subset T_x E)$

such that :

(1.3) $T_x E = H_x E \oplus V_x E$

Having the above theorems we can cuantify the general parallel transport , using the last results related to the vector bundle theories and the existence of the remarcable linear connections $\{D\}$, called d-linear-connections .

Definition 1 ([M]). A linear connection D, on E , which preserve , by parallelism , the horizontal and the vertical distributions H and V on E is called d- linear – connection.

We obtain :

Theorem 1.3. The generalised parallel transport implies the existence of a parallel transport related to a linear connection D , which preserve the horizontal and the vertical distribution H and V .

Having the above results we will be able to introduce in a cosmological modelling , especially in the Einstein Theory of the Generalised Relativity , the notions of horizontal horizont H and vertical horizont V. We have also a calculus algorithm [5].

C. Because in a cosmological modelling is also involved the light phenomena , if we use a pseudo-riemannian metric g(in a particular case having the Lorentz signature), we will be constrained to use those linear connection which preserve the isotropic distributions at the parallel transport.In this way we will reach at the theory of the pairs of the g-conjugated linear connections , elaborated in [4] , [5], [7], [9].

These are few topics proposed by the authors in cosmological modelling and especially in the Generalised Relativity. We will reconsider , in the folowing statements , those aspects .

§2. Cosmological modelling

Every cosmological model is based on the fundamental notion of differentiable manifold M , endowed with a pseudo-riemannian metric g .

It is already known:

Proposition . (2.1.). Let us consider a paracompact , connected , C^{∞}-diferentiable manifold M_n . Then there exists a riemannian metric G on M.

We can not state the same thing regarding the existence of a pseudo-riemannian metric g. In this case we need supplementary conditions , involving the topology , such as the vanishing of the topological invariant Euler-Poincare characteristic (see [14]).

From geometrical point of view this is equivalent with the existence of a vector field $V \in X(M)$, which is a nonzero one in every point.

It results:

Proposition. (2.2). In the hypothesis of the proposition 2.1 , if there exists a vector field $V \in X(M)$ which is a nonzero one in every point then there will exists a pseudo-riemannian metric , having the Lorentz signature.

Proof.: In the hypothesis of the proposition 2.1 globally there exists a riemannian metric G. Having the vector field $V \in X(M)$ which is a nonzero one in every point we will be able to define a tensorial field of type $(0,2)$ on M by :

(2.1) $g(X,Y) = -2G(X,V)G(Y,V) + G(X,Y) \qquad \forall X,Y \in X(M)$

where V has the property $G(V,V) = 1$ in every point $x \in M$.

It is sufficient to consider in $x \in M, (X_1 ... X_{n-1}, V)$ an orthonormal base; it results :

(2.2) $g(X_r, X_r) = +1, r = \overline{1,n}\ ,\ g(V,V) = -1$

Because M is a connected manifold the signature of g is a constant one. So g is a pseudo-riemannian metric having the Lorentz signature .

In the folowing statements we will consider a pseudo-riemannian manifold (M_n, g) in the above conditions for M (so g has the Lorentz signature).

The model from the Einstein Genelised Theory of Relativity will be denoted by $E_\nabla = (M; g; \nabla, \mathcal{T})$, where ∇ is the Levi-Civitta connection defined by the pseudo-riemannian metric g , i.e.

(2.3) $\nabla_X g = 0; \overset{\nabla}{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$

(∇ is a metric connection $\nabla g = 0$ and a symmetrical one , i.e. it is a nulltorsion one $\overset{\nabla}{T} = 0$), and $\mathcal{T}$ is the energy-momentum tensor from the Einstein equations :

(2.4) $\overset{\nabla}{r}(XY) - \frac{1}{2} r g(XY) + \Lambda g(XY) = k\mathcal{T}(XY)$

where Λ is a cosmological constant, k is the Einstein constant and $\overset{\nabla}{r}$ is the Ricci's tensor.

If the value of Λ is very small then we will have the Einstein's equations with the conservation law

(2.5) $div_\nabla \mathcal{T} = 0$

We have to study the change of the linear connection ∇ into another linear connection D in the following cases:

a) D is a g-metrical one $(Dg = 0)$ but with torsion

(2.6) $T(X,Y) = D_X Y - D_Y X - [X,Y] \neq 0$

b) D is a symetrical one $(T=0)$ but it is not a g-metrical one $(Dg \neq 0)$

c) D has torsion $(T \neq 0)$ and it is not a g-metrical one $(Dg \neq 0)$

In the case $T \neq 0$ the model will be called an assymetrical one .

The study of the assymetrical model was , for a long time , a serious problem even for Einstein . In the last part of his life ([1]) he accepted a non-ssymetrical model, by choosing a linear connection D (with $T \neq 0$) which preserve the Einstein's equations and also preserve the law

(2.7) $div_D \mathcal{T} = 0$

having the form

(2.8) $D_X Y = \nabla_X Y + \sigma(X)Y$

where

(2.9) $\sigma = df \qquad (f \in \mathcal{T}(M))$

i.e. σ is an exact 1-form $(\sigma \in \Lambda_1(M))$.

The study in a local map was already done. We have studied in this paper those aspects using a modern , invariant method.

Obviously we have :

(2.10) $d\sigma = 0$ (where d is the external differential)

i.e. σ is a closed form..

Globally , if we have (2.10) then we won't have (2.9); it is only locally true , according to the Poincaré Lemma .

We get:

Theorem (2.3). The assymetric Einstein model can be generalised if we don't have anymore the condition $\sigma = df$.

Notice 1. For D from (2.8) we have $T \neq 0, Dg \neq 0$. That means that we are in a particular case (C).

For a bigger generalisation , useful in every cosmological model (which involve the light phenomena) we will start from the general geometrical study , according to the purpose from (C) (§1).

§3. g-conjugated models

We will denote by $L_1 = (M, g; D_1), L_2 = (M, g; D_2)$, two models for the same cosmological phenomena.

Definition . (3.1). We will say that L_1, L_2 are g-conjugated models if , for every two orthogonal one –dimensional distributions $\overset{(1)}{\mathcal{D}}_1, \overset{(1)}{\mathcal{D}}_2$ on M

(3.1) $g\left(\underset{1}{V}, \underset{2}{V}\right) = 0 \qquad \forall \underset{(1)}{V} \in \overset{(1)}{\mathcal{D}}_1, \underset{2}{V} \in \overset{(1)}{\mathcal{D}}_2$

at their parallel transport , related to $\overset{(1)}{D}, \overset{(2)}{D}$ it is preserved the relation (3.1).

A comprehensiv mathematical study is given in [6] ,[7] ,[9].

Starting from the definition (3.1) we obtain :

Definition. (3.2). Two cosmological models L_1, L_2 will be called g-echivalent ones if the relation ~ on the set $\left\{\left(\overset{(1)}{D}, \overset{(2)}{D}\right)\right\}$ is a relation of equivalence. We will denote in these conditions $L_1 \overset{g}{\sim} L_2$.

We get :

Theorem (3.1). Let us consider two cosmological models L_1, L_2 which are g-conjugated ones .We have $L_1 \overset{g}{\sim} L_2$ if and only if $\overset{(1)}{D}, \overset{(2)}{D}$ are coparallel linear connections , namely they have the same paralellism of the one-dimensional distributions.

Theorem (3.2). Let us consider two cosmological models L_1, L_2 which are g-conjugated ones .We have $L_1 \overset{g}{\sim} L_2$ if and only if we have :

(3.2) $\overset{(2)}{D}_X Y = \overset{(1)}{D}_X Y + \sigma(X) Y \qquad \forall X, Y \in x(M), \sigma \in \Lambda_1(M)$

It results , as a particular case:

Theorem (3.3). The non-symmetrical Einstein model $L_D = (M, g, D)$ is g-equivalent with the simmetrical Einstein model $L_\nabla = (M, g, \nabla)$.

Definition (3.3). A non-symmetrical model $L_D = (M, g, D)$ is strictly equivalent with the simmetric Einstein model L_∇ , if it preserve the Einstein equations :

(3.3) $\overset{(D)}{r}(XY) - \frac{1}{2}\overset{D}{r} = k_D \mathcal{T}_D$

and the conservation law

(3.4) $div_D \mathcal{T}_D = 0$

We will denote in this case $L_D \sim L_\nabla$

It results :

Theorem (3.4). We have $L_D \sim L_\nabla$ if and only if :

(3.5) $D_X Y = \nabla_X Y + \sigma(X)Y$

Where σ is a closed form $(d\sigma = 0)$.

Corollary (3.1). As a particular case the non-symmetrical Einstein model is strictly equivalent with with the simmetrical Einstein model because $d(df) = 0$.

In this way we have a geometrical interpretation of the choice (2.8) (2.9) made by Einstein and also a generalisation of the non-symmetrical model of Einstein .

Therefore every non-symmetrical model L_D of the Theory of the Generalised Relativity which preserve the Einstein equations must have the linear connection D which is coparallel with ∇ , with the closed 1-form σ (in a particular case an exact one). **Corolarry (3.2).** Any non-symmetrical Einstein models $L_1 = \left(M, g, \overset{(1)}{D}\right), L_2 = \left(M, g, \overset{(2)}{D}\right)$ of the Theory of the Generalised Relativity have $\overset{(1)}{D}, \overset{(2)}{D}$ which are coparallel ones , with a closed 1-form.

Both $\overset{(1)}{D}$ and $\overset{(2)}{D}$ can not be compatible with g(namely can not be metrical ones).

In the general case if we consider $E = TM$ and g on E, we will have a decomposition (1.3) and linear d-connections D on $E = TM$ satisfying:

(3.6) $hD_X vY = 0, vD_X hY = 0 \qquad \forall X, Y \in x(E)$

where h, v are the horizontal projector and the vertical projector

(3.7) $g(hX, vY) = 0 \qquad \forall X, Y \in x(E)$

It results :

Theorem (3.5). Two d-linear connections $\overset{(1)}{D}, \overset{(2)}{D}$ on E are g-conjugated ones if and only of they are g-conjugated ones for the one-dimensional distributions

$\left(\overset{(1)}{\underset{1}{D}}{}^h, \overset{(1)}{\underset{2}{D}}{}^h\right)\left(\overset{(1)}{\underset{1}{D}}{}^v, \overset{(1)}{\underset{2}{D}}{}^v\right)$ ($\overset{(1)}{\underset{1}{D}}{}^h$ is the horizontal one-dimensional distribution and $\overset{(1)}{\underset{1}{D}}{}^v$ is the vertical one-dimensional distribution).

Notice 1. The condition (3.1) is verified by itself if $\overset{(1)}{\mathcal{D}}_1 = \overset{(1)}{\underset{1}{\mathcal{D}}}{}^h, \overset{(1)}{\mathcal{D}}_2 = \overset{(1)}{\underset{2}{\mathcal{D}}}{}^v$.

Notice 2. In this case on $E = TM$ there exists an almost complex structure F defined by

(3.8) $FX^h = -X^v; FY^v = Y^h$

where Z^h denote the horizontal lift of $X \in x(M)$ and Z^v denote the vertical lift of $Z \in x(M)$. It results $F^2 = -I$. This structure will be a complex one if it will be an integrable one (i.e. the Nijenhuis's tensor vanishes : $N_F = 0$).

Having these structures we will be able to write the g-conjugation condition into a local , adapted basis etc.

§4. Generalizations

Let us consider a non-symmetrical cosmological model $L_D = (M, g, D)$ which is strictly equivalent with the Einstein model $L_\nabla = (M, g, \nabla)$.

Let us consider two cosmological models $L_{D1} = \left(M, g, D^{(1)}\right)$, $L_{D2} = \left(M, g, D^{(2)}\right)$; let us take only the general condition $L_{D_1} \sim L_{D_2}$ (i.e. the two models to be g-conjugated ones), but this is regarded as a consequence of $L_{D_1} \overset{g}{\sim} L_\nabla$, as the only one connection beetwen these models.

We can write the general transformations:

(4.1) $D_X^{(1)} Y = \nabla_X Y + \tau^{(1)}(X, Y)$

(4.2) $D_X^{(2)} Y = \nabla_X Y + \tau^{(2)}(X, Y)$

$$\forall X, Y \in x(M)$$

where :

(4.3) $\tau^{(1)}, \tau^{(2)}$ are two tensors of type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (the tensors of the affine deformation).

We have (3.5).

Let us impose the condition $L_{D_1} \sim L_{D_2}$. It results:

(4.4) $\alpha(X)g(Y,Z) = \left(\overset{(1)}{D}_X \, g\right)(Y,Z) - g\left(Y, \overset{(21)}{\tau}(X,Z)\right)$

(4.5) $\alpha(X)g(Y,Z) = \left(\overset{(2)}{D}_X \, g\right)(Y,Z) - g\left(Y, \overset{(21)}{\tau}(X,Z)\right)$

$X,Y,Z \in x(M), \alpha \in \Lambda_1(M)$

(4.6) $\overset{(21)}{\tau}(X,Z) = \overset{(2)}{D}_X \, Z - \overset{(1)}{D}_X \, Z$

If we are tacking account of (3.5) we get :

Theorem (4.1). We have :

(4.7) $\overset{(21)}{\tau}(XZ) = \sigma(X)_{\bullet} Z + \overset{(2)}{\tau}(X,Z) - \overset{(1)}{\tau}(X,Z)$

(4.8) $d\sigma = 0$

(4.9) $\beta(X)g(YZ) = \left(\overset{(1)}{D}_X \, g\right)(Y,Z) - g\left(Y, \overset{(2)}{\tau}(X,Z)\right) + g\left(Y, \overset{(1)}{\tau}(X,Z)\right)$

(4.10) $\gamma(X)g(Y,Z) = \left(\overset{(2)}{D}_X \, g\right)(Y,Z) + g\left(Y, \overset{(2)}{\tau}(X,Z)\right) - g\left(Y, \overset{(1)}{\tau}(X,Z)\right)$

where :

(4.11) $\beta = \alpha + \sigma$

(4.12) $\gamma = \alpha - \sigma; d\sigma = 0$

From these relations we get:

Theorem (4.2). L_{D_1}, L_{D_2} are g-conjugated models if and only if :

(4.13) $g\left(\overset{(1)}{\tau}(X,Y), Z\right) + g\left(Y, \overset{(2)}{\tau}(XZ)\right) = \rho(X)g(Y,Z)$

$\forall X,Y,Z \in x(M); \rho \in \Lambda_1(M)$

As a particular case if we impose the supplementary condition that the two models to be g-equivalent ones it results :

Theorem (4.3). If $L_{D_1} \overset{g}{\sim} L_{D_2}$ then we will have :

(4.14) $\overset{(2)}{\tau}(X,Y) = \overset{(1)}{\tau}(X,Y) + (\beta - \sigma)(X)_{\bullet} Y$

where $\beta \in \Lambda_1(M)$ and $\overset{(1)}{\tau}, \overset{(2)}{\tau}$ are satisfaying :

(4.15) $g\left(\overset{1}{\tau}(X,Y), Z\right) + g\left(Y, \overset{(1)}{\tau}(XZ)\right) = (\rho - \beta + \sigma)(X) g(Y,Z)$

(4.16) $g\left(\overset{(2)}{\tau}(X,Y), Z\right) + g\left(Y, \overset{(2)}{\tau}(X,Z)\right) = (\rho + \beta - \sigma)(X) g(Y)$

but also converselly.

Of course we can consider another particular cases.

Our opinion is that we will be able to give a comprehensive model for cosmological theories if we will replace the tangent bundle , $E = TM$, with a vector bundle having the total space E , with the type-fiber R and making a decomposition

(4.17) $TE = HE \oplus VE$

by the condition :

(4.18) $g(hX, vY) = 0, X, Y \in x(E)$

i.e.

(4.19) $g = hg + vg$

where the metric g has Lorentz signature.

We can assume that:

(4.20) hg

is pozitive defined.

A study of the non-symmetrical Einstein models , which are equivalents with L_{∇}, on such kind of structures , is in progress.

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